COVERING ALGEBRAS I: EXTENDED AFFINE LIE ALGEBRAS

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This is the first, of what will be a sequence of three papers, dealing with a generalization of certain parts of the beautiful work of V. Kac on finite order automorphisms of finite dimensional complex simple Lie algebras. Recall that Kac (see [K2, Chapter 8] and [H, §X.5]) built a Lie algebra from a pair $(\mathfrak{g}, \sigma)$ comprised of a finite order automorphism σ of a finite dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ algebra as follows. First from σ he obtains the eigenspaces

$$\mathfrak{g}_{\bar{\imath}} = \{x \in \mathfrak{g} \mid \sigma(x) = \zeta^i x\},$$

where m is the order of σ, $\zeta = e^{2\pi i/m}$, $i \in \mathbb{Z}$ and $i \to \bar{\imath}$ is the natural map of $\mathbb{Z} \to \mathbb{Z}_m$ (here $\mathbb{Z}_m$ denotes the integers modulo m). He then constructs the Lie algebra

$$\operatorname{Aff}(\mathfrak{g}, \sigma) := \Big(\bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\bar{\imath}} \otimes t^i\Big) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where c is central, $d = t\frac{d}{dt}$ is the degree derivation so that $[d, x \otimes t^p] = px \otimes t^p$, and

$$[x \otimes t^p, y \otimes t^q] = [x, y] \otimes t^{p+q} + p(x, y)\delta_{p+q,0}c$$

Here $x \in \mathfrak{g}_{\bar{p}}$, $y \in \mathfrak{g}_{\bar{q}}$ and $p, q \in \mathbb{Z}$, while $(\ ,\ )$ denotes the Killing form of $\mathfrak{g}$. It is known that all affine Kac-Moody Lie algebras arise this way. When σ is the identity, one gets the untwisted affine algebra corresponding to $\mathfrak{g}$ (see [K1] and [M]), whereas the other graph automorphisms, if they exist, lead to the twisted versions of these algebras (see [K1]). By taking into account the conjugacy theorem of Peterson and Kac [PK], one can say that in fact the above construction establishes a bijective correspondence between affine Kac-Moody Lie algebras and conjugacy classes of graph automorphisms of the finite dimensional simple Lie algebras. Furthermore,

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the complex Lie algebra $\mathrm{Aff}(\mathfrak{g},\sigma)$ depends only on the outer part of σ. Thus the $\mathrm{Aff}(\mathfrak{g},\sigma)$'s, which at first appear to make up a very large class of Lie algebras, give up to isomorphism exactly the affine Lie algebras. Kac goes on to use this information to get a classification of all finite order automorphisms of $\mathfrak{g}$.

Our goal in this sequence of three papers is to get, as complete as is possible, a picture of what all of the algebras $\mathrm{Aff}(\mathfrak{g},\sigma)$ look like for other classes of Lie algebras $\mathfrak{g}$ as explained below. But first a word on terminology. At different stages of our work we will find it convenient, and necessary, to also work with several other versions of the algebras $\mathrm{Aff}(\mathfrak{g},\sigma)$. Thus we will work with the "loop version" with no central element and no derivation added. It is given by

$$\mathrm{L}(\mathfrak{g},\sigma) = \bigoplus_{i\in\mathbb{Z}} g_{\bar{\imath}} \otimes t^i$$

with multiplication coordinate wise. We will also need to study the derived algebra of $\mathrm{Aff}(\mathfrak{g},\sigma)$. Of course we need to carefully distinguish these and will do so when necessary. However, we also think of them all simply as *covering algebras* of our original algebra $\mathfrak{g}$ relative to the automorphism σ. Thus, we use the term covering algebra loosely and will develop other specific names for our various algebras when necessary. For example, we will call the algebra $\mathrm{Aff}(\mathfrak{g},\sigma)$ *the affinization of* $\mathfrak{g}$ *relative to the automorphism* σ.

What follows is a very brief account of the topics discussed in the three papers.

1. *Extended affine Lie algebras (EALA's).* These algebras are natural generalizations of affine Kac-Moody Lie algebras, and are the subject of study in this our first paper. EALA's come equipped with analogues of Cartan subalgebras, root systems, invariant forms, etc. The role played by the null roots in the affine Kac-Moody case, is that of the so called isotropic roots. These generate a lattice whose rank, is referred to as the nullity of the EALA. In fact, the result of Kac's theorem in the language of EALA's reads as follows: If $\mathfrak{g}$ is a tame EALA of nullity zero then $\mathrm{Aff}(\mathfrak{g},\sigma)$ is a tame EALA of nullity one and moreover, all such algebras arise in this way. When phrased this way it becomes quite natural to ask what happens in the case of EALA's of higher nullity. In the first paper we look at when the affinization $\mathrm{Aff}(\mathfrak{g},\sigma)$ of an EALA is itself an EALA, and determine the relationship between the corresponding root systems.

2. *Symmetrizable Kac-Moody Lie algebras.* The second paper begins with a general construction that provides a cohomological description of covering algebras by means of forms (a non-abelian H^1). The main thrust of the work lies in trying to decide if in the symmetrizable Kac-Moody case, the algebras $\mathrm{Aff}(\mathfrak{g},\sigma)$ depend only on the outer part of σ. (As we have seen, this is exactly what happens if $\mathfrak{g}$ is finite dimensional.) To study this question, we use the Gantmacher-like description of automorphisms provided by [KW].

3. *Affine Kac-Moody Lie algebras.* In the final paper we study in detail the finer

structure of the EALA's that are obtained as affinizations of the affine Kac-Moody Lie algebras.

Each of the papers will have its own introduction.

§1 Introduction: Basics on EALA's and an outline of results

In this introduction to the present paper, we begin by recalling the definition of an extended affine Lie algebra (EALA for short) and some of the basic properties of these algebras. We will conclude with a short outline of the main results of the paper.

EALA's were introduced by R. Høegh-Krohn and B. Torresani in [H-KT]. Many of the basic facts about EALA's and their root systems were proved in [AABGP]. The reader can consult that reference for any results stated in this section without proof.

Throughout this paper we will work with Lie algebras over the field of complex numbers $\mathbb{C}$. The basic definition of an EALA is broken down into a sequence of axioms EA1–EA4, EA5a, and EA5b.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. We assume first of all that $\mathfrak{g}$ satisfies the following axioms EA1 and EA2.

EA1. $\mathfrak{g}$ has a non-degenerate invariant symmetric bilinear form denoted by

$$(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}.$$

EA2. $\mathfrak{g}$ has a nonzero finite dimensional abelian subalgebra $\mathfrak{h}$ such that $\mathrm{ad}_{\mathfrak{g}} h$ is diagonalizable for all $h \in \mathfrak{h}$ and such that $\mathfrak{h}$ equals its own centralizer, $C_{\mathfrak{g}}(\mathfrak{h})$, in $\mathfrak{g}$.

One lets $\mathfrak{h}^*$ denote the dual space of $\mathfrak{h}$ and for $\alpha \in \mathfrak{h}^*$ we let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Then we have that $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha$ and so since $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$ by EA2, we obtain that $\mathfrak{h} = \mathfrak{g}_0$.

We next define the *root system* R of $\mathfrak{g}$ relative to $\mathfrak{h}$ by saying

$$R = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}.$$

Notice that $0 \in R$ and that R is an extended affine root system (EARS for short) in the sense of [AABGP]. As usual one finds that $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha + \beta = 0$. Thus, $-R = R$ and we also have that $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha + \beta = 0$. In particular, the form is nondegenerate when restricted to $\mathfrak{h} \times \mathfrak{h}$ and so this allows us to transfer the form to $\mathfrak{h}^*$ as follows. For each $\alpha \in \mathfrak{h}^*$ we let t_α be the unique element in $\mathfrak{h}$ satisfying $(t_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$. Then for $\alpha, \beta \in \mathfrak{h}^*$ we let (α, β) be defined by the equation

$$(\alpha, \beta) = (t_\alpha, t_\beta). \tag{1.1}$$

We now have a nondegenerate form on $\mathfrak{h}^*$ and so can speak of isotropic and nonisotropic roots. We let R^0 be the set of isotropic roots and let $R^\times$ be the nonisotropic roots so that we have the disjoint union

$$R = R^0 \cup R^\times.$$

We can now state the remaining axioms.

EA3. For any $\alpha \in R^\times$ and any $x \in \mathfrak{g}_\alpha$ the transformation $\mathrm{ad}_{\mathfrak{g}} x$ is a locally nilpotent on $\mathfrak{g}$.

EA4. R is a discrete subspace of $\mathfrak{h}^*$.

EA5a. $R^\times$ cannot be decomposed into a union $R^\times = R_1 \cup R_2$ where R_1 and R_2 are nonempty orthogonal subsets of $R^\times$.

EA5b. For any $\delta \in R^0$ there is some $\alpha \in R^\times$ such that $\alpha + \delta \in R$.

Definition 1.2. A triple $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ consisting of a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ and a bilinear form $(\cdot,\cdot)$ satisfying EA1-EA4, EA5a, and EA5b is called an *extended affine Lie algebra* or EALA for short.

We often will abuse notation and simply say "Let $\mathfrak{g}$ be an EALA" but the reader should always recognize that we have in mind a fixed triple $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$.

Until further notice we let $\mathfrak{g}$ denote an EALA. We next recall some properties of such algebras.

There are always nonisotropic roots as is shown by applying EA5b to the root 0 (which is a root by EA2). We have $(R^0, R^\times) = \{0\}$. Let V be the real span of R. Then one knows (see Theorem 2.14 of [AABGP]) that the form on $\mathfrak{g}$ can be scaled in such a way that $(\alpha, \beta) \in \mathbb{R}$ for all $\alpha, \beta \in R$ and the form is positive semidefinite on V. Let V^0 be the radical of V. Then the integer $\nu = \dim_{\mathbb{R}} V^0$ is called the *nullity* of the EALA $\mathfrak{g}$. We have that V^0 is the real span of R^0. We denote the natural map from V to $\bar{V} = V/V^0$ by $x \mapsto \bar{x}$, and we let $\bar{R}$ be the image of R in $\bar{V}$. Then $\bar{R}$ is a finite irreducible root system in the space $\bar{V}$ where we use the positive definite form on $\bar{V}$ induced from the semidefinite form on V. ($\bar{R}$ contains 0 and is possibly nonreduced.) The *type* of $\mathfrak{g}$ is by definition the type of the root system $\bar{R}$.

It is also often useful to know that $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in R^\times$.

Definition 1.3. The *core* of the EALA $\mathfrak{g}$, denoted by $\mathfrak{g}_c$, is the subalgebra of $\mathfrak{g}$ generated by the root spaces $\mathfrak{g}_\alpha$ for nonisotropic roots $\alpha \in R^\times$.

It is easy to see that the core $\mathfrak{g}_c$ of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$.

Definition 1.4. We say the EALA $\mathfrak{g}$ is *tame* if the centralizer $C_{\mathfrak{g}}(\mathfrak{g}_c)$ of $\mathfrak{g}_c$ in $\mathfrak{g}$ is contained in $\mathfrak{g}_c$. Equivalently $\mathfrak{g}$ is tame if $C_{\mathfrak{g}}(\mathfrak{g}_c)$ equals the center $Z(\mathfrak{g}_c)$ of $\mathfrak{g}_c$.

We are now in a position to briefly describe the contents of this paper.

In Section 2 we give the general definition of $\mathrm{L}(\mathfrak{g}, \sigma)$ and $\mathrm{Aff}(\mathfrak{g}, \sigma)$. For the definition of $\mathrm{L}(\mathfrak{g}, \sigma)$ all that is required is a Lie algebra $\mathfrak{g}$ and a finite order automorphism σ of $\mathfrak{g}$. To define $\mathrm{Aff}(\mathfrak{g}, \sigma)$ one requires in addition a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. As already mentioned, we call $\mathrm{Aff}(\mathfrak{g}, \sigma)$ the affinization

of $\mathfrak{g}$ relative to σ. $\mathrm{Aff}(\mathfrak{g},\sigma)$ can be realized as the subalgebra of fixed points of an automorphism of the Lie algebra $\mathrm{Aff}(\mathfrak{g},\mathrm{id}_{\mathfrak{g}})$ (which is just the usual affinization of $\mathfrak{g}$.) It is always the case that $\mathrm{Aff}(\mathfrak{g},\sigma)$ has a nondegenerate symmetric bilinear form which is invariant.

Section 3 contains the main results of this paper. We work with an EALA $(\mathfrak{g},\mathfrak{h},(\cdot,\cdot))$ and an automorphism σ of $\mathfrak{g}$ which satisfies the following four properties:

A1. $\sigma^m = 1$.

A2. $\sigma(\mathfrak{h}) = \mathfrak{h}$.

A3. $(\sigma(x),\sigma(y)) = (x,y)$ for all $x,y\in\mathfrak{g}$.

A4. The centralizer of $\mathfrak{h}^\sigma$ in $\mathfrak{g}^\sigma$ equals $\mathfrak{h}^\sigma$.

Here we have let $\mathfrak{h}^\sigma$ (respectively $\mathfrak{g}^\sigma$) denote the fixed points of σ in $\mathfrak{h}$ (respectively $\mathfrak{g}$).

Assuming A1–A4, $\mathrm{Aff}(\mathfrak{g},\sigma)$ has a natural choice of a finite dimensional abelian ad-diagonalizable subalgebra, namely $(\mathfrak{h}^\sigma\otimes 1)\oplus\mathbb{C}c\oplus\mathbb{C}d$. Also $\mathrm{Aff}(\mathfrak{g},\sigma)$ has a natural nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)$ which is the form defined in section 2. We investigate whether or not the triple $(\mathrm{Aff}(\mathfrak{g},\sigma),(\mathfrak{h}^\sigma\otimes 1)\oplus\mathbb{C}c\oplus\mathbb{C}d,(\cdot,\cdot))$ is an EALA by investigating whether axioms EA1, EA2, EA3, EA4, EA5a and EA5b hold. We find that the first five of these always hold. In fact we see, in Proposition 3.25, that A4 is precisely the assumption needed in order to obtain EA2.

Dealing with axiom EA5b, as well as whether or not $\mathrm{Aff}(\mathfrak{g},\sigma)$ is tame, is less straightforward. Our main result, Theorem 3.63, states that if $\mathfrak{g}$ is tame then either $\mathrm{Aff}(\mathfrak{g},\sigma)$ has no nonisotropic roots or is again a tame EALA. It also describes how to tell which of these (mutually exclusive) alternatives holds using only information about the root system R of $\mathfrak{g}$ and the action of σ on R. Thus, we give a very general procedure for producing new tame EALA's from old ones. This is accomplished without using the classification of tame EALA's that is currently being developed (see [BGK], [BGKN], [Y] and [AG]).

In the case when $\mathrm{Aff}(\mathfrak{g},\sigma)$ is a tame EALA, we also obtain in Section 3 a description of the root system, core, type and nullity of $\mathrm{Aff}(\mathfrak{g},\sigma)$ in terms of the corresponding objects for $\mathfrak{g}$. We further show that if $\mathfrak{g}$ is non-degenerate (the definition of this term for EALA's is recalled when we need it), then $\mathrm{Aff}(\mathfrak{g},\sigma)$ is also nondegenerate.

As a byproduct of our investigation in Section 3, we notice a general fact about the axioms of an EALA. Namely EA5b follows from the other axioms of an EALA together with tameness. This may be of some independent interest.

Finally, in Section 4 we present 3 examples of how to use our main result to obtain EALA's using affinization. In particular, in the third example we look at the important case when $\mathfrak{g}$ is taken to be an affine Kac-Moody Lie algebra and σ is a diagram automorphism. A1, A2 and A3 hold by definition of σ and we show A4 also holds in all cases. We find further that $\mathrm{Aff}(\mathfrak{g},\sigma)$ has a nonisotropic root if and only if the diagram automorphism is not transitive on the nodes of the diagram. It follows from our results that $\mathrm{Aff}(\mathfrak{g},\sigma)$ is a tame EALA of nullity 2 in all cases but one. The one exception is when $\mathfrak{g}$ is of type $A_l^{(1)}$ and σ is a graph automorphism

which is transitive on the set of nodes of the diagram. In the present paper we leave our investigation of covering algebras of affine algebras here even though we fully realize there is the natural question of identifying these algebras. This question will be investigated in the third paper in this series. We note also that covering algebras of affine algebras have already been studied in [H-KT], [W] and [Po].

§2 The Lie algebras $\mathrm{L}(\mathfrak{g},\sigma)$ and $\mathrm{Aff}(\mathfrak{g},\sigma)$

In this section, we give the general definitions of the Lie algebras $\mathrm{L}(\mathfrak{g},\sigma)$ and $\mathrm{Aff}(\mathfrak{g},\sigma)$.

Throughout the section we assume that $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$. (Although we work here with a complex Lie algebra, the reader will notice that all that is really required in this section is that the base field contain a primitive m^{th} root of unity, where m is a period for the automorphisms being considered.)

Definition 2.1. The *loop algebra* of $\mathfrak{g}$ is the Lie algebra $\mathrm{L}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

Suppose next that m is a positive integer and σ is an automorphism of $\mathfrak{g}$ of period m. Notice that we do not require that m is the actual order of σ but only a period of σ. To define $\mathrm{L}(\mathfrak{g},\sigma)$, we need some notation. Let

$$\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$$

be the group of integers mod m and let $i \to \bar{\imath}$ denote the natural homomorphism of $\mathbb{Z}$ onto $\mathbb{Z}_m$. Finally, let $\zeta = e^{2\pi i/m}$. Then $\mathfrak{g}$ can be decomposed into eigenspaces for σ as

$$\mathfrak{g} = \bigoplus_{i=0}^{m-1} \mathfrak{g}_{\bar{\imath}}, \quad \text{where } g_{\bar{\imath}} = \{x \in \mathfrak{g} \mid \sigma(x) = \zeta^i x\}.$$

Observe that $\mathfrak{g}_{\bar{0}}$ is the subalgebra $\mathfrak{g}^\sigma$ of fixed points of σ in $\mathfrak{g}$.

Now σ extends to an automorphism (also denoted by σ) of $\mathrm{L}(\mathfrak{g})$ defined by

$$\sigma(x \otimes t^i) = \zeta^{-i}\sigma(x) \otimes t^i.$$

The subalgebra $\mathrm{L}(\mathfrak{g})^\sigma$ of fixed points of σ in $\mathrm{L}(\mathfrak{g})$ is equal to $\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{\imath}} \otimes t^i$.

Definition 2.2. Assume that σ is an automorphism of $\mathfrak{g}$ such that $\sigma^m = 1$. Let $\mathrm{L}(\mathfrak{g},\sigma,m)$ be the Lie algebra $\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{\imath}} \otimes t^i$. It is easy to see that $\mathrm{L}(\mathfrak{g},\sigma,m) \cong \mathrm{L}(\mathfrak{g},\sigma,n)$ if we have $\sigma^m = 1 = \sigma^n$ and so, up to isomorphism, this algebra does not depend on the period m. Thus, we drop the dependence on m and just write $\mathrm{L}(\mathfrak{g},\sigma)$ for $\mathrm{L}(\mathfrak{g},\sigma,m)$. So we have

$$\mathrm{L}(\mathfrak{g},\sigma) = \mathrm{L}(\mathfrak{g})^\sigma = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{\imath}} \otimes t^i$$

and $\mathrm{L}(\mathfrak{g},\sigma)$ is a subalgebra of the loop algebra $\mathrm{L}(\mathfrak{g}) = \mathrm{L}(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}})$. We call $\mathrm{L}(\mathfrak{g},\sigma)$ the *loop algebra of* $\mathfrak{g}$ *relative to* σ.

If $0 \neq a \in \mathbb{C}$, then the map $x \otimes t^k \mapsto a^k x$ is a Lie algebra homomorphism of $\mathrm{L}(\mathfrak{g})$ onto $\mathfrak{g}$. The restriction of this homomorphism maps $\mathrm{L}(\mathfrak{g},\sigma)$ onto $\mathfrak{g}$. This is the reason why $\mathrm{L}(\mathfrak{g},\sigma)$ (along with other related algebras) is regarded as a covering algebra of $\mathfrak{g}$ (see [H, §X.5]).

To discuss affinization, we assume that $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$ with a nondegenerate invariant symmetric bilinear form

$$(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}.$$

Definition 2.3. The *affinization* of $\mathfrak{g}$ is defined as

$$\mathrm{Aff}(\mathfrak{g}) = \big(\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]\big) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Multiplication on $\mathrm{Aff}(\mathfrak{g})$ is defined by

$$[x\otimes t^i + r_1c + r_2d, y\otimes t^j + s_1c + s_2d] = [x,y]\otimes t^{i+j} + jr_2 y\otimes t^j - is_2 x\otimes t^i + i\delta_{i+j,0}(x,y)c,$$

where $x,y \in \mathfrak{g}, i,j \in \mathbb{Z}, r_1,r_2,s_1,s_2 \in \mathbb{C}$. One easily checks that $\mathrm{Aff}(\mathfrak{g})$ is a Lie algebra. We extend the form on $\mathfrak{g}$ to one on $\mathrm{Aff}(\mathfrak{g})$ by decreeing that

$$(x\otimes t^i + r_1c + r_2d, y\otimes t^j + s_1c + s_2d) = \delta_{i+j,0}(x,y) + r_1s_2 + r_2s_1, \tag{2.4}$$

and see immediately that this gives a nondegenerate invariant symmetric bilinear form on $\mathrm{Aff}(\mathfrak{g})$.

Note that the above construction coincides with the usual affinization in the case when $\mathfrak{g}$ is a finite dimensional simple Lie algebra. Also, c is central in $\mathrm{Aff}(\mathfrak{g})$ and d is just the usual degree derivation of $\mathbb{C}[t,t^{-1}]$ lifted to this affinization.

To define relative affinization, we need the following lemma.

Lemma 2.5. *Let σ be an automorphism of $\mathfrak{g}$ such that $\sigma^m = 1$ and $(\sigma x,\sigma y) = (x,y)$ for all $x,y \in \mathfrak{g}$. Then σ extends to an automorphism (also denoted by σ) of $\mathrm{Aff}(\mathfrak{g})$ defined by*

$$\sigma(x\otimes t^i + rc + sd) = \zeta^{-i}\sigma(x)\otimes t^i + rc + sd. \tag{2.6}$$

This extension preserves the extended form 2.4 on $\mathrm{Aff}(\mathfrak{g})$, has period m and fixes c and d. The subalgebra $\mathrm{Aff}(\mathfrak{g})^\sigma$ of fixed points σ in $\mathrm{Aff}(\mathfrak{g})$ is given by

$$\mathrm{Aff}(\mathfrak{g})^\sigma = \Big(\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{i}} \otimes t^i\Big) + \mathbb{C}c + \mathbb{C}d.$$

Consequently, $(\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{i}} \otimes t^i) + \mathbb{C}c + \mathbb{C}d$ is a subalgebra of $\mathrm{Aff}(\mathfrak{g})$ and the bilinear form on $\mathrm{Aff}(\mathfrak{g})$ is nondegenerate when restricted to this subalgebra.

Proof. All of the statements here are easily checked. $\square$

Definition 2.7. Assume that σ is an automorphism of $\mathfrak{g}$ such that $\sigma^m = 1$ and that $(\sigma x, \sigma y) = (x, y)$ for all $x, y \in \mathfrak{g}$. Let $\mathrm{Aff}(\mathfrak{g}, \sigma, m)$ be the Lie algebra $(\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{i}} \otimes t^i) + \mathbb{C}c + \mathbb{C}d$. If we have $\sigma^m = 1 = \sigma^n$, it is easy to see that there is a form preserving isomorphism of $\mathrm{Aff}(\mathfrak{g}, \sigma, m)$ onto $\mathrm{Aff}(\mathfrak{g}, \sigma, n)$. So $\mathrm{Aff}(\mathfrak{g}, \sigma)$ does not depend on the period m, and we just write $\mathrm{Aff}(\mathfrak{g}, \sigma)$ for $\mathrm{Aff}(\mathfrak{g}, \sigma, m)$. Thus we have

$$\mathrm{Aff}(\mathfrak{g}, \sigma) = \mathrm{Aff}(\mathfrak{g})^\sigma = (\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{i}} \otimes t^i) + \mathbb{C}c + \mathbb{C}d$$

and $\mathrm{Aff}(\mathfrak{g}, \sigma)$ is a subalgebra of the affinization $\mathrm{Aff}(\mathfrak{g}) = \mathrm{Aff}(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}})$. The form 2.4 restricted to $\mathrm{Aff}(\mathfrak{g}, \sigma)$ is a nondegenerate invariant symmetric bilinear form on $\mathrm{Aff}(\mathfrak{g}, \sigma)$. We call $\mathrm{Aff}(\mathfrak{g}, \sigma)$ the *affinization of* $\mathfrak{g}$ *relative to* σ.

§3 New EALA's from Old

In this section we investigate the conditions required for the affinization of an EALA relative to an automorphism of finite order to be an EALA. We begin by specifying the assumptions that will be in force throughout the section.

Basic Assumption 3.1. Assume that $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ is an EALA, $m \in \mathbb{Z}$, $m \geq 1$ and σ is automorphism of $\mathfrak{g}$ which satisfies

A1. $\sigma^m = 1$.
A2. $\sigma(\mathfrak{h}) = \mathfrak{h}$.
A3. $(\sigma(x), \sigma(y)) = (x, y)$ for all $x, y \in \mathfrak{g}$.

Later we will add one more assumption, A4, on σ (see Basic Assumption 3.27).

Since we are mainly interested in the algebra $\mathrm{Aff}(\mathfrak{g}, \sigma)$, we make notation easier by simply writing

$$\tilde{\mathfrak{g}} = \mathrm{Aff}(\mathfrak{g}, \sigma) = \big(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\big) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

We identify $\mathfrak{g} = \mathfrak{g} \otimes 1$ as a subalgebra of $\tilde{\mathfrak{g}}$, and we set

$$\tilde{\mathfrak{h}} = \mathfrak{h}^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

$\tilde{\mathfrak{h}}$ is an abelian subalgebra of $\tilde{\mathfrak{g}}$. Also, we let $(\cdot,\cdot)$ denote the form on $\tilde{\mathfrak{g}}$ obtained by restricting the form 2.4 from $\mathrm{Aff}(\mathfrak{g})$ to $\tilde{\mathfrak{g}}$.

Then specifically our problem in this section is to find conditions under which the triple $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ is an EALA.

Using Lemma 2.5 we have the following:

Lemma 3.2. *The form $(\cdot,\cdot)$ on $\tilde{\mathfrak{g}}$ is a nondegenerate invariant symmetric bilinear form. Hence, $\tilde{\mathfrak{g}}$ satisfies EA1.*

We next wish to consider EA2. Thus, we need to look at the adjoint action of $\tilde{\mathfrak{h}}$ on $\tilde{\mathfrak{g}}$. To do this we will need to look at the adjoint action of $\mathfrak{h}^\sigma$ on $\mathfrak{g}$. As in Section 1, R will denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Also, for any vector space X over $\mathbb{C}$, we let X^* denote the dual space of X over $\mathbb{C}$.

Definition 3.3. With notation as above we let $\pi : \mathfrak{h}^* \to (\mathfrak{h}^\sigma)^*$ be the linear map defined by saying $\pi(\alpha)$ is just α restricted to $\mathfrak{h}^\sigma$. Symbolically, $\pi(\alpha) = \alpha|_{\mathfrak{h}^\sigma}$.

This mapping π will play an important role in what follows. When we consider $\mathfrak{h}^\sigma$ acting on $\mathfrak{g}$ via the adjoint action we see that each element acts semisimply so that the set of weights for this action is just $\pi(R)$. Furthermore, the eigenspace $\mathfrak{g}_{\bar{\imath}}$ for $\bar{\imath} \in \mathbb{Z}_m$ is stabilized by $\mathfrak{h}^\sigma$ so we let $\mathfrak{g}_{\bar{\imath},\pi(\alpha)}$ be the $\pi(\alpha)$ weight space of $\mathfrak{h}^\sigma$ acting on $\mathfrak{g}_{\bar{\imath}}$. That is for all $\alpha \in R$ and all $\bar{\imath} \in \mathbb{Z}_m$ we have

$$\mathfrak{g}_{\bar{\imath},\pi(\alpha)} = \{x \in \mathfrak{g}_{\bar{\imath}} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}^\sigma\}. \tag{3.4}$$

We certainly have

$$\mathfrak{g}_{\bar{\imath}} = \bigoplus_{\pi(\alpha)\in\pi(R)} \mathfrak{g}_{\bar{\imath},\pi(\alpha)} \text{ for any } i \in \mathbb{Z}.$$

Next, we let

$$R_{\bar{\imath}} = \{\alpha \in R \mid \mathfrak{g}_{\bar{\imath},\pi(\alpha)} \neq (0)\} \text{ for any } i \in \mathbb{Z}. \tag{3.5}$$

Then we obtain

$$\mathfrak{g}_{\bar{\imath}} = \bigoplus_{\pi(\alpha)\in\pi(R_{\bar{\imath}})} \mathfrak{g}_{\bar{\imath},\pi(\alpha)} \text{ for } i \in \mathbb{Z} \text{ and so} \tag{3.6}$$

$$\mathfrak{g} = \bigoplus_{i=0}^{m-1} \bigoplus_{\pi(\alpha)\in\pi(R_{\bar{\imath}})} \mathfrak{g}_{\bar{\imath},\pi(\alpha)}. \tag{3.7}$$

Notice that we have if $\alpha, \beta \in R$ satisfy $\pi(\alpha) = \pi(\beta)$ then $\alpha \in R_{\bar{\imath}}$ if and only if $\beta \in R_{\bar{\imath}}$ for any $i \in \mathbb{Z}$. Moreover, for $\alpha \in R$ we have that

$$\mathfrak{g}_\alpha \subseteq \bigoplus_{i=0}^{m-1} \mathfrak{g}_{\bar{\imath},\pi(\alpha)}, \tag{3.8}$$

which implies that $\alpha \in R_{\bar{\imath}}$ for some $i \in \mathbb{Z}$. Thus, we have the following union which is not necessarily disjoint:

$$R = \bigcup_{i=0}^{m-1} R_{\bar{\imath}}. \tag{3.9}$$

We clearly have for $\alpha, \beta \in R, i, j \in \mathbb{Z}$ that either $[\mathfrak{g}_{\bar{\imath},\pi(\alpha)}, \mathfrak{g}_{\bar{\jmath},\pi(\beta)}] = (0)$ or there is some $\gamma \in R_{\bar{\imath}+\bar{\jmath}}$ with $\pi(\alpha) + \pi(\beta) = \pi(\gamma)$ and

$$[\mathfrak{g}_{\bar{\imath},\pi(\alpha)}, \mathfrak{g}_{\bar{\jmath},\pi(\beta)}] \subseteq \mathfrak{g}_{\bar{\imath}+\bar{\jmath},\pi(\gamma)}. \tag{3.10}$$

Next, we record how the spaces $\mathfrak{g}_{\bar{\imath},\pi(\alpha)}$ interact with our form. For $i, j \in \mathbb{Z}$ and $\alpha, \beta \in R$, if either $\bar{\imath} + \bar{\jmath} \neq 0$ or if $\pi(\alpha) + \pi(\beta) \neq 0$ then we have

$$(\mathfrak{g}_{\bar{\imath},\pi(\alpha)}, \mathfrak{g}_{\bar{\jmath},\pi(\beta)}) = (0). \tag{3.11}$$

Since the form is nondegenerate it follows that $-\alpha \in R_{-\bar{\imath}}$ and the form pairs the spaces $\mathfrak{g}_{\bar{\imath},\pi(\alpha)}$ and $\mathfrak{g}_{-\bar{\imath},\pi(-\alpha)}$ in a nondegenerate fashion.

Since we are assuming that $\sigma(\mathfrak{h}) = \mathfrak{h}$ we can decompose $\mathfrak{h}$ relative to σ. This gives us

$$\mathfrak{h} = \bigoplus_{i=0}^{m-1} \mathfrak{h}_{\bar{\imath}} \text{ and } \mathfrak{h}_{\bar{0}} = \mathfrak{h}^{\sigma}. \tag{3.12}$$

Also, because σ preserves the form we find that $(\mathfrak{h}_{\bar{\imath}}, \mathfrak{h}_{\bar{\jmath}}) = 0$ whenever $\bar{\imath} + \bar{\jmath} \neq 0$. It follows that we have

$$(\mathfrak{h}^{\sigma})^{\perp} = \bigoplus_{i=1}^{m-1} \mathfrak{h}_{\bar{\imath}}. \tag{3.13}$$

We therefore have

$$\mathfrak{h} = \mathfrak{h}^{\sigma} \oplus (\mathfrak{h}^{\sigma})^{\perp} \quad \text{and correspondingly} \quad \mathfrak{h}^* = (\mathfrak{h}^{\sigma})^* \oplus ((\mathfrak{h}^{\sigma})^{\perp})^*,$$

where we are identifying $(\mathfrak{h}^{\sigma})^*$ and $((\mathfrak{h}^{\sigma})^{\perp})^*$ inside of $\mathfrak{h}^*$ by letting any element from $(\mathfrak{h}^{\sigma})^*$ act as zero on $(\mathfrak{h}^{\sigma})^{\perp}$ and letting any element from $((\mathfrak{h}^{\sigma})^{\perp})^*$ act as zero on $\mathfrak{h}^{\sigma}$. With these identifications, note that $\pi : \mathfrak{h}^* \to (\mathfrak{h}^{\sigma})^*$ is just the projection onto the first factor in the sum $\mathfrak{h}^* = (\mathfrak{h}^{\sigma})^* \oplus ((\mathfrak{h}^{\sigma})^{\perp})^*$.

We let σ acts on $\mathfrak{h}^*$ by saying that

$$\sigma(\alpha)(h) = \alpha(\sigma^{-1}(h)) \text{ for all } h \in \mathfrak{h}. \tag{3.14}$$

This allows us to consider $(\mathfrak{h}^*)^{\sigma}$, the fixed points of σ in $\mathfrak{h}^*$. It is not hard to see that $(\mathfrak{h}^*)^{\sigma} = (\mathfrak{h}^{\sigma})^*$ and that $((\mathfrak{h}^{\sigma})^{\perp})^* = ((\mathfrak{h}^*)^{\sigma})^{\perp}$ and that this latter space is nothing but the sum of the eigenspaces of σ in $\mathfrak{h}^*$ corresponding to eigenvalues different from 1. For the convenience of the reader we next state this as a lemma and sketch a proof.

Lemma 3.15. *With notation as above we have that*

$$(\mathfrak{h}^*)^{\sigma} = (\mathfrak{h}^{\sigma})^* \quad \textit{and} \quad ((\mathfrak{h}^{\sigma})^{\perp})^* = ((\mathfrak{h}^*)^{\sigma})^{\perp}$$

with this latter space being the sum of the eigenspaces of σ in $\mathfrak{h}^*$ corresponding to eigenvalues different from 1.

Proof. We just prove the first statement as the second one is similar. We let $\alpha \in (\mathfrak{h}^{\sigma})^* \subseteq \mathfrak{h}^*$ so that α is zero on $(\mathfrak{h}^{\sigma})^{\perp}$. For $h \in \mathfrak{h}^{\sigma}$ we have $\sigma(h) = h$ so that

$\sigma^{-1}(h) = h$ and so by 3.14 we get that $\sigma(\alpha)(h) = \alpha(h)$ so both α and $\sigma(\alpha)$ agree on $\mathfrak{h}^\sigma$. If $h \in (\mathfrak{h}^\sigma)^\perp$ then because σ preserves the form so does σ^{-1} and so it follows that σ^{-1} maps $(\mathfrak{h}^\sigma)^\perp$ to itself. Thus $\alpha(\sigma^{-1}(h)) = 0$ and this implies that $\sigma(\alpha)(h) = 0$ from which it follows that both $\sigma(\alpha)$ and α agree on the space $(\mathfrak{h}^\sigma)^\perp$. This gives us $(\mathfrak{h}^\sigma)^* \subseteq (\mathfrak{h}^*)^\sigma$.

Conversely, let $\alpha \in (\mathfrak{h}^*)^\sigma$. Then α and $\sigma(\alpha)$ agree on all of $\mathfrak{h}$. Next, recall 3.13 and choose $\bar{\imath} \in \mathbb{Z}_m \setminus \{0\}$, as well as an element $h \in \mathfrak{h}_{\bar{\imath}}$, so that $\sigma(h) = \zeta^i h$ and hence $\sigma^{-1}(h) = \zeta^{-i} h$. Thus, $\alpha(h) = \sigma(\alpha)(h) = \zeta^{-i}\alpha(h)$ so since $\zeta^{-i} \neq 1$ we obtain that $\alpha(h) = 0$.Thus α and $\sigma(\alpha)$ are both zero on $(\mathfrak{h}^\sigma)^\perp$ and so we now have $\alpha = \sigma(\alpha)$ is in $(\mathfrak{h}^\sigma)^*$ as desired. $\square$

We are now ready to calculate the adjoint action of $\tilde{\mathfrak{h}}$ on $\tilde{\mathfrak{g}}$. Since

$$\tilde{\mathfrak{h}} = \mathfrak{h}^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

we can identify $(\mathfrak{h}^\sigma)^*$ as a subspace of $\tilde{\mathfrak{h}}^*$ so that the elements of $(\mathfrak{h}^\sigma)^*$ act as zero on both c and d. We also define elements γ and δ in $\tilde{\mathfrak{h}}^*$ by saying

$$\gamma(\mathfrak{h}^\sigma) = \{0\} = \delta(\mathfrak{h}^\sigma), \quad \gamma(c) = 1 = \delta(d), \quad \gamma(d) = 0 = \delta(c). \tag{3.16}$$

Thus, we have that

$$\tilde{\mathfrak{h}}^* = (\mathfrak{h}^\sigma)^* \oplus \mathbb{C}\gamma \oplus \mathbb{C}\delta. \tag{3.17}$$

It is clear that $\tilde{\mathfrak{h}}$ acts semisimply on $\tilde{\mathfrak{g}}$ via the adjoint representation. Thus, if we set $\tilde{\mathfrak{g}}_{\tilde{\alpha}} = \{x \in \tilde{\mathfrak{g}} \mid [h,x] = \tilde{\alpha}(h)x \text{ for all } h \in \tilde{\mathfrak{h}}\}$ for all $\tilde{\alpha} \in \tilde{\mathfrak{h}}^*$, we have $\tilde{\mathfrak{g}} = \bigoplus_{\tilde{\alpha}\in\tilde{\mathfrak{g}}^*} \tilde{\mathfrak{g}}_{\tilde{\alpha}}$. We set

$$\tilde{R} = \{\tilde{\alpha} \in \mathfrak{h}^* \mid \mathfrak{g}_{\tilde{\alpha}} \neq 0\},$$

the set of roots of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}}$. Then,

$$\tilde{\mathfrak{g}} = \bigoplus_{\tilde{\alpha}\in\tilde{R}} \tilde{\mathfrak{g}}_{\tilde{\alpha}}. \tag{3.18}$$

Next 3.4 and 3.5 imply that if $R_{\bar{\imath}}$ is nonempty then, for any $i \in \mathbb{Z}$, $\mathfrak{g}_{\bar{\imath},\pi(\alpha)} \otimes t^i$ is decomposable into weight spaces for $\tilde{\mathfrak{h}}$ corresponding to weights from $\pi(R_{\bar{\imath}}) + i\delta$. If $R_{\bar{\imath}}$ is empty we take $\pi(R_{\bar{\imath}}) + i\delta$ to be the empty set for any $i \in \mathbb{Z}$. We thus have

$$\tilde{R} = \bigcup_{i\in\mathbb{Z}} (\pi(R_{\bar{\imath}}) + i\delta) \subseteq \tilde{\mathfrak{h}}^*. \tag{3.19}$$

Moreover, if $i \in \mathbb{Z}$, $\alpha \in R_{\bar{\imath}}$ and $\pi(\alpha) + i\delta \neq 0$ then

$$\tilde{\mathfrak{g}}_{\pi(\alpha)+i\delta} = \mathfrak{g}_{\bar{\imath},\pi(\alpha)} \otimes t^i. \tag{3.20}$$

On the other hand, we have that

$$\tilde{\mathfrak{g}}_0 = (\mathfrak{g}_{\bar{0},0} \otimes 1) \oplus \mathbb{C}c \oplus \mathbb{C}d. \tag{3.21}$$

But from 3.4 we see that $\mathfrak{g}_{\bar{0},0}$ is nothing but the centralizer of $\mathfrak{h}^\sigma$ in $\mathfrak{g}^\sigma$. That is,

$$\tilde{\mathfrak{g}}_0 = C_{\mathfrak{g}^\sigma}(\mathfrak{h}^\sigma) \oplus \mathbb{C}c \oplus \mathbb{C}d. \tag{3.22}$$

Before stating our result about EA2, we need a result about our map π defined in 3.3. This is provided in the following lemma.

Lemma 3.23. *For $\alpha \in \mathfrak{h}^*$ we have that $\pi(\alpha) = \frac{1}{m}\sum_{i=0}^{m-1} \sigma^i(\alpha)$.*

Proof. For $\alpha \in \mathfrak{h}^*$ write $\alpha = \alpha_{\bar{0}} + \cdots + \alpha_{\bar{m}-\bar{1}}$ where $\alpha_{\bar{\imath}} \in (\mathfrak{h}^*)_{\bar{\imath}}$. Then we have $\pi(\alpha) = \alpha_{\bar{0}}$. But $\frac{1}{m}\sum_{i=0}^{m-1} \sigma^i(\alpha) = \alpha_{\bar{0}} + \frac{1}{m}\sum_{i=0}^{m-1} \sigma^i(\sum_{\bar{\jmath}\in\mathbb{Z}_m\setminus\{0\}} \alpha_{\bar{\jmath}})$. Thus it is enough to show that $\sum_{i=0}^{m-1} \sigma^i(\alpha_{\bar{\jmath}}) = 0$ for $\bar{\jmath} \in \mathbb{Z}_m \setminus \{0\}$. But we have $\sum_{i=0}^{m-1} \sigma^i(\alpha_{\bar{\jmath}}) = \sum_{i=0}^{m-1} \zeta^{ij}\alpha_{\bar{\jmath}} = \frac{(\zeta^j)^m - 1}{\zeta^j - 1}\alpha_{\bar{\jmath}}$, which clearly equals zero. $\square$

We need one more definition before stating our result about EA2. To prepare for this notice that for $\alpha \in R$ we have $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}$ and hence

$$\sigma(R) = R.$$

Definition 3.24. For $\alpha \in R$ we let $\ell_\sigma(\alpha)$ denote the smallest positive integer satisfying $\sigma^{\ell_\sigma(\alpha)}(\alpha) = \alpha$. We call $\ell_\sigma(\alpha)$ the σ *length* of α.

For a root $\alpha \in R$ we have that $\alpha, \sigma(\alpha), \dots, \sigma^{\ell_\sigma(\alpha)-1}(\alpha)$ are distinct. Moreover $\ell_\sigma(\alpha)$ divides the period m.

Proposition 3.25. *The following statements are equivalent.*

(i) *The centralizer of $\tilde{\mathfrak{h}}$ in $\tilde{\mathfrak{g}}$ equals $\tilde{\mathfrak{h}}$. In other words, EA2 holds for $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{h}}$.*
(ii) *The centralizer of $\mathfrak{h}^\sigma$ in $\mathfrak{g}^\sigma$ equals $\mathfrak{h}^\sigma$.*
(iii) *If $\alpha \in R \setminus \{0\}$ then either $\pi(\alpha) \neq 0$ or $\{x \in \mathfrak{g}_a \mid \sigma^{\ell_\sigma(\alpha)}(x) = x\} = \{0\}$.*

Moreover, if m is prime these statements are equivalent to the following.

(iv) *If $\alpha \in R$ and if $\alpha \neq 0$ then $\pi(\alpha) \neq 0$.*

Proof. From 3.22 we see that (i) and (ii) are equivalent. We first show that (ii) implies (iii). Thus, assume (ii) holds but that (iii) fails. Then for some nonzero $\alpha \in R$ and some nonzero $x \in \mathfrak{g}_\alpha$ we have $\pi(\alpha) = 0$ and $\sigma^{\ell_\sigma(\alpha)}(x) = x$. We write ℓ for $\ell_\sigma(\alpha)$ and let $y = x + \sigma(x) + \cdots + \sigma^{\ell-1}(x)$. Now $y \neq 0$ as the roots $\alpha, \sigma(\alpha), \dots, \sigma^{\ell-1}(\alpha)$ are distinct and we have $\sigma(y) = y$ so that $y \in \mathfrak{g}^\sigma$. Since $\pi(\alpha) = 0$ we have that $\alpha(\mathfrak{h}^\sigma) = \{0\}$. Thus for $i = 0, 1, \dots, \ell - 1$ we have $(\sigma^i(\alpha))(\mathfrak{h}^\sigma) = \{0\}$ and hence $[\mathfrak{h}^\sigma, y] = \{0\}$. That is $y \in C_{\mathfrak{g}^\sigma}(\mathfrak{h}^\sigma)$. Since $\alpha \neq 0$, we have $y \notin \mathfrak{h}$ and so $y \notin \mathfrak{h}^\sigma$. This contradicts (ii).

For the converse we assume now that (iii) holds but that (ii) fails. Thus, there exists some $x \in C_{\mathfrak{g}^\sigma}(\mathfrak{h}^\sigma)$ so that $x \notin \mathfrak{h}^\sigma$. We can write x as $x = \sum_{\alpha\in R} x_\alpha$ where $x_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in R$. Since $\sigma(x) = x$ it follows that $\sigma^k(x_\alpha) = x_{\sigma^k(\alpha)}$ for $\alpha \in R$, $k \in \mathbb{Z}$, $k \geq 0$. Thus $\sigma^{\ell_\sigma(\alpha)}(x_\alpha) = x_\alpha$ for $\alpha \in R$. Also $[\mathfrak{h}^\sigma, x] = \{0\}$ so that $[\mathfrak{h}^\sigma, x_\alpha] = \{0\}$ for $\alpha \in R$. Thus $\alpha(\mathfrak{h}^\sigma)x_\alpha = \{0\}$ for $\alpha \in R$. Hence, if $\alpha \in R$ and $x_\alpha \neq 0$ then $\alpha(\mathfrak{h}^\sigma) = \{0\}$ and so $\pi(\alpha) = 0$. From (iii) it follows that $x_\alpha = 0$ for all $\alpha \in R \setminus \{0\}$ and so we obtain that $x = x_0 \in \mathfrak{g}_0 = \mathfrak{h}$. This gives us the contradiction that $x \in \mathfrak{h}^\sigma$ so establishes that (iii) implies (ii).

Finally suppose that m is prime. Clearly we only need to show (iii) implies (iv) which we do by contradiction. Thus, assume there exists some $\alpha \in R \setminus \{0\}$ such that $\pi(\alpha) = 0$. By Lemma 3.23 we have that $\sigma(\alpha) \neq \alpha$ and hence $\ell_\sigma(\alpha) \neq 1$. But

$\ell_\sigma(\alpha)$ divides m which is prime and so $\ell_\sigma(\alpha) = m$. Hence $\sigma^{\ell_\sigma(\alpha)}(x_\alpha) = x_\alpha$ for all $x_\alpha \in \mathfrak{g}_\alpha$ and this contradicts (iii). $\square$

The conditions in the proposition automatically hold if for any $\alpha \in R \setminus \{0\}$ we have $\pi(\alpha) \neq 0$. Thus we have the following corollary.

Corollary 3.26. *If for any $\alpha \in R \setminus \{0\}$ we have $\pi(\alpha) \neq 0$ then $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, \tilde{R})$ satisfies EA2.*

Because of Proposition 3.25 we see that it is important for us that the automorphism σ of $\mathfrak{g}$ satisfies $C_{\mathfrak{g}^\sigma}(\mathfrak{h}^\sigma) = \mathfrak{h}^\sigma$. We will require this as an assumption from now on.

Basic Assumption 3.27. We now assume that the automorphism σ of the EALA $\mathfrak{g}$ satisfies A1, A2, A3 and

A4. The centralizer of $\mathfrak{h}^\sigma$ in $\mathfrak{g}^\sigma$ equals $\mathfrak{h}^\sigma$.

Remark 3.28. Because of this basic assumption we see from Lemma 3.2 and Proposition 3.25 that $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ satisfies EA1 and EA2.

We now go on to consider the axioms EA3, EA4, EA5a and EA5b for $\tilde{\mathfrak{g}}$.

Recall that R^0 and $R^\times$ denote the set of isotropic and nonisotropic roots respectively in R. Similarly we let $\tilde{R}^0$ and $\tilde{R}^\times$ denote the set of isotropic and nonisotropic roots respectively of $\tilde{R}$.

From 3.16 and 3.17 we have that

$$(\gamma,\gamma) = 0 = (\delta,\delta) \quad \text{and} \quad (\gamma,\delta) = 1. \tag{3.29}$$

This follows immediately from the way in which we transfer the form from $\tilde{\mathfrak{h}}$ to $\tilde{\mathfrak{h}}^*$. The following fact, which is easy to see, should be noted. The restriction of the form on $\tilde{\mathfrak{h}}$ to $\mathfrak{h}^\sigma$ is the same as the restriction of the form from $\mathfrak{h}$ to $\mathfrak{h}^\sigma$. It follows also that the restriction of the form from $\tilde{\mathfrak{h}}^*$ to $(\mathfrak{h}^\sigma)^*$ equals the restriction of the form from $\mathfrak{h}^*$ to $(\mathfrak{h}^\sigma)^*$.

Now 3.19 says that $\tilde{R} = \bigcup_{i\in\mathbb{Z}}(\pi(R_{\bar{\imath}}) + i\delta)$. Also for $i \in \mathbb{Z}$, $\alpha \in R_{\bar{\imath}}$ we have $(\pi(\alpha)+i\delta, \pi(\alpha)+i\delta) = (\pi(\alpha), \pi(\alpha))$ where now the right hand side can be interpreted in $\mathfrak{h}^*$ or in $\tilde{\mathfrak{h}}^*$. Thus we obtain that

$$\tilde{R}^\times = \bigcup_{i\in\mathbb{Z}} \{\pi(\alpha) + i\delta \mid \alpha \in R_{\bar{\imath}},\ (\pi(\alpha),\pi(\alpha)) \neq 0\} \quad \text{and} \tag{3.30}$$

$$\tilde{R}^0 = \bigcup_{i\in\mathbb{Z}} \{\pi(\alpha) + i\delta \mid \alpha \in R_{\bar{\imath}},\ (\pi(\alpha),\pi(\alpha)) = 0\}. \tag{3.31}$$

Lemma 3.32. *Let $\tilde{\alpha} \in \tilde{R}^\times$ and $x \in \tilde{\mathfrak{g}}_{\tilde{\alpha}}$. Then the operator $ad_{\tilde{\mathfrak{g}}}(x)$ is nilpotent. Thus $\tilde{\mathfrak{g}}$ satisfies EA3.*

Proof. We have that $\tilde{\alpha} = \pi(\alpha) + i\delta$ where $i \in \mathbb{Z}$, $\alpha \in R_{\bar{\imath}}$ and $(\pi(\alpha),\pi(\alpha)) \neq 0$. Then $x \in \mathfrak{g}_{\bar{\imath},\pi(\alpha)} \otimes t^i$ and we put $h = t_{\pi(\alpha)} \in \mathfrak{h}^\sigma$ (recall 1.1). Then $(\pi(\alpha))(h) =$

$(\pi(\alpha), \pi(\alpha))$ and so $[h, x] = (\pi(\alpha), \pi(\alpha))x$. Hence x is an eigenvector for $\mathrm{ad}(h)$ corresponding to a non-zero eigenvalue. Thus, to show that $\mathrm{ad}_{\tilde{\mathfrak{g}}}(x)$ is nilpotent it suffices to show that the diagonalizable operator $\mathrm{ad}_{\tilde{\mathfrak{g}}}(h)$ has only finitely many eigenvalues.

By 3.19 the eigenvalues of $\mathrm{ad}_{\tilde{\mathfrak{g}}}(h)$ are the scalars of the form $(\pi(\epsilon) + j\delta)(h)$ where $j \in \mathbb{Z}$, $\epsilon \in R_{\bar{j}}$. But since $h = t_{\pi(\alpha)} \in \mathfrak{h}^\sigma$ we have $(\pi(\epsilon) + j\delta)(h) = (\pi(\epsilon))(h) = (\pi(\epsilon), \pi(\alpha)) = (\epsilon, \pi(\alpha))$. By Lemma 3.23 we see this equals $\frac{1}{m}(\epsilon, \sum_{i=0}^{m-1} \sigma^i(\alpha))$. Thus, each eigenvalue of $\mathrm{ad}_{\tilde{\mathfrak{g}}}(h)$ is in the set $\frac{1}{m}(A + \cdots + A)$ (m summands), where $A = \{(\epsilon, \nu) \mid \epsilon, \nu \in R\}$. But we know that the set A is finite (see 2.11 and 2.14 in Chapter 1 and 2.10 in Chapter 2 of [AABGP]). $\square$

Lemma 3.33. *$\tilde{R}$ is a closed and discrete subspace of $\tilde{\mathfrak{h}}^*$. In particular, $\tilde{\mathfrak{g}}$ satisfies EA4.*

Proof. We must show that any sequence in $\tilde{R}$ which converges to an element of $\tilde{\mathfrak{h}}^*$ is eventually constant. Looking at 3.19 we see that such a sequence must eventually have the $i\delta$-part constant. So it is enough for us to show that for fixed i any sequence in $\pi(R_{\bar{i}})$ which converges to an element of $(\mathfrak{h}^\sigma)^*$ is eventually constant. Thus, it suffices to show that any sequence $\{\pi(\alpha_n)\}_{1 \le n < \infty}$ in $\pi(R)$ which converges to an element of $\mathfrak{h}^*$ is eventually constant. Now by Lemma 3.23 together with the fact that $\sigma(R) = R$ we get that $\pi(R) \subseteq \frac{1}{m}(R + \cdots + R) \subseteq \frac{1}{m}\langle R \rangle$ where $\langle R \rangle$ denotes the additive subgroup of $\mathfrak{h}^*$ generated by R. Thus $\pi(\alpha_n) \in \frac{1}{m}\langle R \rangle$. But we know from 2.18 in Chapter II of [AABGP] that $\langle R \rangle$ is a lattice in the real span of R. Thus so is $\frac{1}{m}\langle R \rangle$. But the real span of R is closed in $\mathfrak{h}^*$ and so our sequence $\{\pi(\alpha)_n\}_{1 \le n < \infty}$ converges to an element of this span. Hence our sequence is eventually constant as desired. $\square$

We next want to investigate axiom EA5a. In order to do this we will need a lemma about finite irreducible root systems. Since we are using this term in a slightly nonstandard sense, we make our definition precise. A *finite irreducible root system* Ω in a nonzero real Euclidean space X is a finite spanning set for X so that $0 \in \Omega$, $2(\alpha, \beta)(\beta, \beta)^{-1} \in \mathbb{Z}$ for α, β in $\Omega^\times$, $\alpha - 2(\alpha, \beta)(\beta, \beta)^{-1}\beta \in \Omega^\times$ for α, β in $\Omega^\times$, and $\Omega^\times$ cannot be written as the union of two nonempty orthogonal sets, where $\Omega^\times$ denotes the set of nonzero elements of Ω.

Lemma 3.34. *Let Ω be a finite irreducible root system in a nonzero real Euclidean space X. Let Y be a nonzero subspace of X, and let $p : X \to Y$ be the orthogonal projection onto Y. Then*

(i) *If $\alpha \in \Omega^\times$, there exists $\beta \in \Omega^\times$ so that $p(\beta) \neq 0$ and $(\alpha, \beta) \neq 0$.*
(ii) *$p(\Omega^\times) \setminus \{0\}$ cannot be written as the union of two nonempty orthogonal sets.*

Proof. It will be convenient to introduce some temporary notation and terminology. Let $\Delta = \Omega^\times$ (and so Δ is a root system in the usual sense). If $\alpha \in \Delta$, we say that α is *visible* if $p(\alpha) \neq 0$. Also, if $\alpha \in \Delta$, we say that α is *nearly visible* if there exists $\beta \in \Delta$ so that β is visible and $(\alpha, \beta) \neq 0$. We let Δ_{v} (resp. Δ_{nv}) be the set of visible

(resp. nearly visible) roots of Δ. Note that Δ_{v} is a subset of Δ_{nv}. Also, since Δ spans X and since $Y \neq \{0\}$, we have $p(\Delta) \neq \{0\}$ and hence Δ_{v} is nonempty.

We now claim that

$$\alpha \in \Delta_{\mathrm{nv}} \setminus \Delta_{\mathrm{v}} \implies \begin{array}{l} \text{There exist } \beta, \gamma \in \Delta_{\mathrm{v}} \text{ such that} \\ \alpha = \gamma - \beta \text{ and } p(\gamma) = p(\beta). \end{array} \tag{3.35}$$

Indeed, suppose that $\alpha \in \Delta_{\mathrm{nv}} \setminus \Delta_{\mathrm{v}}$. Then, $(\alpha, \beta) \neq 0$ for some $\beta \in \Delta_{\mathrm{v}}$. In that case, $\alpha \neq \pm\beta$ and so $\alpha + \beta$ or $\alpha - \beta$ is in Δ. Replacing β by $-\beta$ if needed, we can assume that $\gamma := \alpha + \beta \in \Delta$. Furthermore, $p(\gamma) = p(\alpha) + p(\beta) = p(\beta) \neq 0$ and so $\gamma \in \Delta_{\mathrm{v}}$. This proves 3.35.

Now $\Delta = \Delta_{\mathrm{nv}} \cup (\Delta \setminus \Delta_{\mathrm{nv}})$ and, by 3.35, the sets Δ_{nv} and $\Delta \setminus \Delta_{\mathrm{nv}}$ are orthogonal. But $\Delta_{\mathrm{nv}} \supseteq \Delta_{\mathrm{v}} \neq \emptyset$. Hence, by the irreducibility of Δ we have

$$\Delta = \Delta_{\mathrm{nv}}, \tag{3.36}$$

which is statement (i).

To prove (ii), suppose that

$$p(\Delta_{\mathrm{v}}) = M_1 \cup M_2, \tag{3.37}$$

where M_1 and M_2 are orthogonal (and hence disjoint). Since $p(\Delta) \setminus \{0\} = p(\Delta_{\mathrm{v}})$, it will suffice to prove that M_1 or M_2 is empty.

Now by 3.37, we have the disjoint union

$$\Delta_{\mathrm{v}} = K_1 \cup K_2, \tag{3.38}$$

where $K_i = \{\alpha \in \Delta \mid p(\alpha) \in M_i\}$, $i = 1, 2$. We claim next that

$$K_1 \text{ and } K_2 \text{ are orthogonal.} \tag{3.39}$$

Indeed, suppose for contradiction that there exist $\alpha \in K_1$, $\beta \in K_2$ with $(\alpha, \beta) \neq 0$. Reasoning as above, we can assume that $\alpha + \beta \in \Delta$. But $(p(\alpha), p(\beta)) = 0$ and so $p(\alpha + \beta) \neq 0$. Consequently, $\alpha + \beta \in \Delta_{\mathrm{v}}$ and therefore $\alpha + \beta$ is in K_1 or K_2. We may assume that $\alpha + \beta$ is in K_1. Then, $p(\alpha + \beta) \in M_1$ and $p(\alpha) \in M_1$, which implies that $(p(\beta), p(\beta)) = (p(\alpha + \beta) - p(\alpha), p(\beta) = 0$. This contradiction proves 3.39.

Finally, 3.35, 3.36 and 3.38 tell us that $\Delta = \Psi_1 \cup \Psi_2$, where

$$\Psi_i = K_i \cup \{\gamma - \beta \mid \beta, \gamma \in K_i,\ p(\gamma) = p(\beta),\ \gamma - \beta \in \Delta\},$$

$i = 1, 2$. By 3.39, Ψ_1 and Ψ_2 are orthogonal. Thus, Ψ_1 or Ψ_2 is empty, and so K_1 or K_2 is empty. Hence, M_1 or M_2 is empty as desired. $\square$

We now return to our investigation of the axiom EA5a for $\tilde{\mathfrak{g}}$. We first need to recall some notation from Section 1. Let V denote the real span of R and assume

that the form on $\mathfrak{g}$ is scaled in such a way that $(\alpha, \beta) \in \mathbb{R}$ for all $\alpha, \beta \in R$ and the form is positive semidefinite on V. V^0 denotes the radical of V and the natural map from V to $\bar{V} = V/V^0$ is denoted $x \mapsto \bar{x}$. Let $\bar{R}$ be the image of R in $\bar{V}$. Then $\bar{R}$ is a finite irreducible root system in $\bar{V}$ where we use the positive definite form on $\bar{V}$ induced from the form on V. Notice that the map $x \to \bar{x}$ maps $R^\times$ onto the set $\bar{R}^\times$ of nonzero roots of $\bar{R}$.

From Lemma 3.23 we see that $\pi(R) \subseteq V$ and hence $\pi(V) \subseteq V$. In particular this implies that

$$V = (V \cap (\mathfrak{h}^\sigma)^*) \perp (V \cap ((\mathfrak{h}^\sigma)^\perp)^*). \tag{3.40}$$

Furthermore, since the form is positive semidefinite on V we get that

$$V^0 = (V^0 \cap (\mathfrak{h}^\sigma)^*) \perp (V^0 \cap ((\mathfrak{h}^\sigma)^\perp)^*). \tag{3.41}$$

It follows from this that we have, upon making the obvious identifications, that

$$\bar{V} = \Big((V \cap (\mathfrak{h}^\sigma)^*)/(V^0 \cap (\mathfrak{h}^\sigma)^*)\Big) \perp \Big((V \cap ((\mathfrak{h}^\sigma)^\perp)^*)/(V^0 \cap ((\mathfrak{h}^\sigma)^\perp)^*)\Big). \tag{3.42}$$

Note that 3.41 also implies that $\pi(V^0) \subseteq V^0$, so that π induces a map $\bar{\pi} : \bar{V} \to \bar{V}$. Moreover, it is clear that $\bar{\pi}$ is just the projection onto the first factor in 3.42.

We have seen that $\sigma(R) = R$ and so it follows that $\sigma(V) = V$. Since σ preserves the form we get that $\sigma(V^0) = V^0$ and so σ induces an element $\bar{\sigma}$ of the orthogonal group of our form on $\bar{V}$. Furthermore, we have that

$$(V \cap (\mathfrak{h}^\sigma)^*)/(V^0 \cap (\mathfrak{h}^\sigma)^*) = \bar{V}^{\bar{\sigma}}, \tag{3.43}$$

and $((V \cap ((\mathfrak{h}^\sigma)^\perp)^*)/(V^0 \cap ((\mathfrak{h}^\sigma)^\perp)^*)$ equals the sum of the eigenspaces for $\bar{\sigma}$ corresponding to eigenvalues not equal to 1. Also, it follows from Lemma 3.23 that

$$\bar{\pi}(\bar{\alpha}) = \frac{1}{m} \sum_{i=0}^{m-1} \bar{\sigma}^i(\bar{\alpha})$$

for $\bar{\alpha} \in \bar{V}$.

We let $\tilde{V}$ be the real span of $\tilde{R}$ in $\tilde{\mathfrak{h}}^*$. Recall from 3.19 we have that

$$\tilde{R} = \bigcup_{i=0}^{m-1} (\pi(R_{\bar{i}}) + i\Delta + m\mathbb{Z}\Delta). \tag{3.44}$$

Since $0 \in R$ we have $0 \in R_{\bar{i}}$ for some i and so in this case we obtain that $i\Delta + m\mathbb{Z}\Delta \subseteq \tilde{R}$. Thus $\Delta \in \tilde{V}$ so we obtain that $\pi(R_{\bar{i}}) \subseteq \tilde{V}$ for $0 \le i \le m-1$. Thus $\tilde{V}$ is spanned by Δ together with $\pi(R) = \bigcup_{i=0}^{m-1} \pi(R_{\bar{i}})$. But by 3.40 the real span of $\pi(R)$ is $V \cap (\mathfrak{h}^\sigma)^*$. It follows that

$$\tilde{V} = (V \cap (\mathfrak{h}^\sigma)^*) \oplus \mathbb{R}\Delta = V^\sigma \oplus \mathbb{R}\Delta. \tag{3.45}$$

Next we note that the restriction of the form from $\tilde{\mathfrak{h}}^*$ to $V \cap (\mathfrak{h}^\sigma)^*$ is the same as the restriction of the form from $\mathfrak{h}^*$ to $V \cap (\mathfrak{h}^\sigma)^*$. It follows that the form on $\tilde{\mathfrak{h}}^*$ restricted to $\tilde{V}$ is real valued and positive semidefinite with Δ in its radical. Hence the radical, $\tilde{V}^0$, of the form on $\tilde{V}$ is given by

$$\tilde{V}^0 = (V^0 \cap (\mathfrak{h}^\sigma)^*) \oplus \mathbb{R}\Delta = (V^0)^\sigma \oplus \mathbb{R}\Delta. \tag{3.46}$$

Next we let $\overline{\tilde{V}} = \tilde{V}/\tilde{V}^0$ and let $x \mapsto \bar{x}$ be the associated natural homomorphism. Let $\overline{\tilde{R}}$ be the image of $\tilde{R}$ in $\overline{\tilde{R}}$. Then using 3.45 and 3.46 and making the obvious identifications we have

$$\overline{\tilde{V}} = (V \cap (\mathfrak{h}^\sigma)^*)/(V^0 \cap (\mathfrak{h}^\sigma)^*) = V^\sigma/(V^0)^\sigma. \tag{3.47}$$

In view of 3.42 and 3.43 we can identify $\overline{\tilde{V}}$ with the space $\bar{V}^{\bar{\sigma}}$ of $\bar{\sigma}$ fixed points in $\bar{V}$. Under this identification the element $\overline{\alpha_0 + r\Delta} = \alpha_0 + r\Delta + \tilde{V}^0 \in \overline{\tilde{V}}$ is identified with $\overline{\alpha_0} = \alpha_0 + V^0 \in \bar{V}$ for $\alpha_0 \in V \cap (\mathfrak{h}^\sigma)^*$, $r \in \mathbb{R}$. Hence this identification preserves the forms on $\overline{\tilde{V}}$ and $\bar{V}$. But we have that $\overline{\tilde{R}} = \{\overline{\pi(\alpha) + i\Delta} \mid i \in \mathbb{Z}, \alpha \in R_{\bar{\imath}}\}$, and $\bar{R} = \{\bar{\alpha} \mid i \in \mathbb{Z}, \alpha \in R_{\bar{\imath}}\}$. Hence under our identification

$$\overline{\tilde{R}} = \bar{\pi}(\bar{R}). \tag{3.48}$$

Lemma 3.49. *We have*

(i) *If $\tilde{R}^\times$ is not empty and $\alpha \in R^\times$, then there exists $\beta \in R^\times$ such that $(\pi(\beta), \pi(\beta)) \neq 0$ and $(\alpha, \beta) \neq 0$.*

(ii) *$\tilde{R}^\times$ cannot be written as the union of two nonempty orthogonal subsets. Thus, EA5a holds for $\tilde{\mathfrak{g}}$.*

Proof. To prove both of these statements we can assume that $\tilde{R}^\times$ is nonempty (since (ii) is trivial if $\tilde{R}^\times$ is empty). Then, $(\pi(\alpha), \pi(\alpha)) \neq 0$ for some $\alpha \in R^\times$ by 3.30. Thus, $\bar{\pi}(\bar{\alpha}) \neq 0$ for some $\bar{\alpha} \in \bar{R}^\times$. So, since $\bar{\pi} : \bar{V} \to \bar{V}$ is the orthogonal projection onto $\bar{V}^{\bar{\sigma}}$, we see that $\bar{V}^{\bar{\sigma}}$ is nonzero. Therefore, we can apply Lemma 3.34 to the configuration $\Omega = \bar{R}$, $X = \bar{V}$, $Y = \bar{V}^{\bar{\sigma}}$ and $p = \bar{\pi}$.

To prove (i), let $\alpha \in R^\times$. Then, $\bar{\alpha} \in \bar{R}^\times$. So, by Lemma 3.34 (i), there exists $\bar{\beta} \in \bar{R}^\times$ so that $\bar{\pi}(\bar{\beta}) \neq 0$ and $(\bar{\alpha}, \bar{\beta}) \neq 0$. Then, lifting $\bar{\beta}$ to an element β of $R^\times$, we have $(\pi(\beta), \pi(\beta)) = (\bar{\pi}(\bar{\beta}), \bar{\pi}(\bar{\beta})) \neq 0$ and $(\alpha, \beta) \neq 0$.

To prove (ii), suppose for contradiction that $\tilde{R}^\times = \tilde{R}_1 \cup \tilde{R}_2$, where $\tilde{R}_1$ and $\tilde{R}_2$ are orthogonal nonempty sets. Applying the map $x \mapsto \bar{x}$, we see that $\overline{\tilde{R}} \setminus \{0\}$ is the union of two nonempty orthogonal sets. By 3.48, the same is true for $\bar{\pi}(\bar{R}) \setminus \{0\}$. Since $\bar{\pi}(\bar{R}) \setminus \{0\} = \bar{\pi}(\bar{R}^\times) \setminus \{0\}$, this contradicts Lemma 3.34 (ii). $\square$

We next wish to compare the cores of our Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ (see Lemma 3.57 below). Recall from [AABGP] that for $\alpha \in R^\times$ we have the reflections w_α which satisfy

$$w_\alpha(\beta) = \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha \quad \text{for } \beta \in \mathfrak{h}^*. \tag{3.50}$$

Moreover, choosing $e_\alpha \in \mathfrak{g}_\alpha, e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ for $\alpha \in R^\times$ as in 1.18 of [AABGP] we have the automorphisms (see 1.23 of [AABGP])

$$\theta_\alpha(t) = \exp(\mathrm{ad}(te_\alpha))\exp(\mathrm{ad}(-t^{-1}e_{-\alpha}))\exp(\mathrm{ad}(te_\alpha)),$$

of $\mathfrak{g}$ for any $t \in \mathbb{C}$ which satisfy

$$\theta_\alpha(t)\mathfrak{g}_\beta = \mathfrak{g}_{w_\alpha(\beta)} \text{ for any } \beta \in R. \tag{3.51}$$

We will let θ_α denote the element $\theta_\alpha(1)$ in what follows.

Lemma 3.52. *Suppose that $\tilde{R}^\times$ is not empty. Then the core, $\mathfrak{g}_c$, of $\mathfrak{g}$ is generated as an algebra by the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in R$, $(\pi(\alpha), \pi(\alpha)) \neq 0$.*

Proof. We let $\mathfrak{m}$ be the subalgebra of $\mathfrak{g}$ generated by all the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in R, (\pi(\alpha), \pi(\alpha)) \neq 0$. Then $\mathfrak{m} \subseteq \mathfrak{g}_c$. To prove the reverse inclusion we must show that $\mathfrak{m}$ contains each root space $\mathfrak{g}_\beta$ for $\beta \in R^\times$. If $(\pi(\beta), \pi(\beta)) \neq 0$ this is clear so we assume that $(\pi(\beta), \pi(\beta)) = 0$. By Lemma 3.49 (i) there exists $\alpha \in R^\times$ such that $(\pi(\alpha), \pi(\alpha)) \neq 0$ and $(\alpha, \beta) \neq 0$. Put $\epsilon = w_\alpha(\beta) = \beta - 2((\beta, \alpha)/(\alpha, \alpha))\alpha \in R^\times$. Since $\pi(\beta) \in V^0$ we have that $(\pi(\epsilon), \pi(\epsilon)) = (-2((\beta, \alpha)/(\alpha, \alpha)))^2(\pi(\alpha), \pi(\alpha)) \neq 0$. Then we have (using the notation developed before the statement of our lemma) that $\theta_\alpha(\mathfrak{g}_\epsilon) = \mathfrak{g}_{w_\alpha(\epsilon)} = \mathfrak{g}_\beta$. But we know that $\mathfrak{g}_\epsilon$, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are contained in $\mathfrak{m}$. Hence, so is $\theta_\alpha(\mathfrak{g}_\epsilon)$, and we obtain that $\mathfrak{g}_\beta \subseteq \mathfrak{m}$ as desired. $\square$

Lemma 3.53. *Suppose that $\tilde{R}^\times$ is not empty. Then $\mathfrak{g}_c$ is the sum of the spaces*

$$\mathfrak{g}_{\bar{\imath},\pi(\alpha)} \text{ for } \bar{\imath} \in \mathbb{Z}_m,\ \alpha \in R_{\bar{\imath}},\ (\pi(\alpha), \pi(\alpha)) \neq 0, \tag{3.54}$$

together with their commutators.

Proof. Let $\mathfrak{s}$ be the sum of the spaces in 3.54. We first show that $\mathfrak{s} \subseteq \mathfrak{g}_c$. For this, let $\bar{\imath} \in \mathbb{Z}_m$, $\alpha \in \mathbb{R}_{\bar{\imath}}$, $(\pi(\alpha), \pi(\alpha)) \neq 0$. Then $\mathfrak{g}_{\bar{\imath},\pi(\alpha)}$ is contained in the $\pi(\alpha)$-weight space for $\mathrm{ad}(\mathfrak{h}^\sigma)$. Thus $\mathfrak{g}_{\bar{\imath},\pi(\alpha)} \subseteq \sum g_\beta$ where the sum is over those $\beta \in R$ satisfying $\pi(\beta) = \pi(\alpha)$. But any $\beta \in R$ satisfying $\pi(\beta) = \pi(\alpha)$ also satisfies $(\pi(\beta), \pi(\beta)) \neq 0$ and so is in $R^\times$. Hence $\mathfrak{s} \subseteq \mathfrak{g}_c$.

We next show that $\mathfrak{s}$ generates $\mathfrak{g}_c$ as an algebra. By Lemma 3.52 it is enough to show that $\mathfrak{s}$ contains all root spaces $\mathfrak{g}_\alpha$ with $\alpha \in R$, $(\pi(\alpha), \pi(\alpha)) \neq 0$. But for such an α, $\mathfrak{g}_\alpha$ is contained in the $\pi(\alpha)$-weight space for $\mathrm{ad}(\mathfrak{h}^\sigma)$ and so $\mathfrak{g}_\alpha \subseteq \sum_{i=0}^{m-1} \mathfrak{g}_{\bar{\imath},\pi(\alpha)}$. Hence $\mathfrak{g}_\alpha \subseteq \mathfrak{s}$.

Since $\mathfrak{s}$ generates $\mathfrak{g}_c$ as an algebra it only remains to show that $\mathfrak{s} + [\mathfrak{s}, \mathfrak{s}]$ is closed under the Lie product and to show this it is enough to show that

$$[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subseteq \mathfrak{s} + [\mathfrak{s}, \mathfrak{s}]. \tag{3.55}$$

For this let $i, j, k \in \mathbb{Z}$, $\alpha \in R_{\bar{\imath}}$, $\beta \in R_{\bar{\jmath}}$, $\epsilon \in R_{\bar{k}}$, with $(\pi(\alpha), \pi(\alpha)) \neq 0, (\pi(\beta), \pi(\beta)) \neq 0, (\pi(\epsilon), \pi(\epsilon)) \neq 0$. We want to show that

$$[\mathfrak{g}_{\bar{\imath},\pi(\alpha)}, [\mathfrak{g}_{\bar{\jmath},\pi(\beta)}, \mathfrak{g}_{\bar{k},\pi(\epsilon)}] \subseteq \mathfrak{s} + [\mathfrak{s}, \mathfrak{s}].$$

To prove this we can clearly assume that the left hand side is not zero and so then $[\mathfrak{g}_{\bar{\jmath},\pi(\beta)}, \mathfrak{g}_{\bar{k},\pi(\epsilon)}]$ is also non-zero. Thus by 3.10, there exists some $\eta \in R_{\bar{\jmath}+\bar{k}}$ so that $\pi(\eta) = \pi(\beta) + \pi(\epsilon)$ and $[\mathfrak{g}_{\bar{\jmath},\pi(\beta)}, \mathfrak{g}_{\bar{k},\pi(\epsilon)}] \subseteq \mathfrak{g}_{\bar{\jmath}+\bar{k},\pi(\eta)}$. So it suffices to show that

$$[\mathfrak{g}_{\bar{\imath},\pi(\alpha)}, \mathfrak{g}_{\bar{\jmath}+\bar{k},\pi(\eta)}] \subseteq \mathfrak{s} + [\mathfrak{s}, \mathfrak{s}]. \tag{3.56}$$

If $(\pi(\eta), \pi(\eta)) \neq 0$ then the left hand side in 3.56 is in $[\mathfrak{s}, \mathfrak{s}]$. Thus, we can assume that $(\pi(\eta), \pi(\eta)) = 0$. But there is some $\omega \in R_{\bar{\imath}+\bar{\jmath}+\bar{k}}$ such that $\pi(\omega) = \pi(\alpha) + \pi(\eta)$ and

$$[\mathfrak{g}_{\bar{\imath},\pi(\alpha)} \mathfrak{g}_{\bar{\jmath}+\bar{k},\pi(\eta)}] \subseteq \mathfrak{g}_{\bar{\imath}+\bar{\jmath}+\bar{k},\pi(\omega)}.$$

Also, since $\pi(\eta) \in V^0$ we have $(\pi(\omega), \pi(\omega)) = (\pi(\alpha), \pi(\alpha)) \neq 0$. It follows that we now have $[\mathfrak{g}_{\bar{\imath},\pi(\alpha)}, \mathfrak{g}_{\bar{\jmath}+\bar{k},\pi(\eta)}] \subseteq \mathfrak{s}$ which is what we want. $\square$

Although $\tilde{\mathfrak{g}}$ may not be in general an EALA, we make the same definitions of the core of $\tilde{\mathfrak{g}}$ and tameness of $\tilde{\mathfrak{g}}$ as in Section 1. Thus, we define the *core* of $\tilde{\mathfrak{g}}$ to be the subalgebra $\tilde{\mathfrak{g}}_c$ of $\tilde{\mathfrak{g}}$ generated by $\tilde{\mathfrak{g}}_{\tilde{\alpha}}$, $\tilde{\alpha} \in \tilde{R}^\times$. If $\tilde{R}^\times$ is empty, we take $\tilde{\mathfrak{g}}_c = \{0\}$. We say that $\tilde{\mathfrak{g}}$ is *tame* provided that the centralizer of $\tilde{\mathfrak{g}}_c$ in $\tilde{\mathfrak{g}}$ is contained $\tilde{\mathfrak{g}}_c$. In the next two lemmas, we calculate the core of $\tilde{\mathfrak{g}}$ and give conditions which imply that $\tilde{\mathfrak{g}}$ is tame.

The automorphism σ of $\mathfrak{g}$ stabilizes the core $\mathfrak{g}_c$ of $\mathfrak{g}$ and hence it also stabilizes the center $Z(\mathfrak{g}_c)$ of $\mathfrak{g}_c$. Thus we have

$$\mathfrak{g}_c = \bigoplus_{i=0}^{m-1} (\mathfrak{g}_c)_{\bar{\imath}} \quad \text{and} \quad Z(\mathfrak{g}_c) = \bigoplus_{i=0}^{m-1} Z(\mathfrak{g}_c)_{\bar{\imath}}$$

where $(\mathfrak{g}_c)_{\bar{\imath}} = \mathfrak{g}_{\bar{\imath}} \cap \mathfrak{g}_c$ and $Z(\mathfrak{g}_c)_{\bar{\imath}} = \mathfrak{g}_{\bar{\imath}} \cap Z(\mathfrak{g}_c)$ for $\bar{\imath} \in \mathbb{Z}_m$.

Lemma 3.57. *Suppose $\tilde{R}^\times$ is not empty. Then*

$$\tilde{\mathfrak{g}}_c = \Big(\bigoplus_{i\in\mathbb{Z}} (\mathfrak{g}_c)_{\bar{\imath}} \otimes t^i\Big) \oplus \mathbb{C}c, \tag{3.58}$$

$$Z(\tilde{\mathfrak{g}}_c) = \Big(\bigoplus_{i\in\mathbb{Z}} Z(\mathfrak{g}_c)_{\bar{\imath}} \otimes t^i\Big) \oplus \mathbb{C}c, \tag{3.59}$$

$$\tilde{\mathfrak{g}}_c / Z(\tilde{\mathfrak{g}}_c) \cong \mathrm{L}(\mathfrak{g}_c / Z(\mathfrak{g}_c), \sigma), \tag{3.60}$$

where $\mathrm{L}(\mathfrak{g}_c/Z(\mathfrak{g}_c), \sigma)$ *denotes the loop algebra of* $\mathfrak{g}_c/Z(\mathfrak{g}_c)$ *relative to the automorphism induced by* σ *on* $\mathfrak{g}_c/Z(\mathfrak{g}_c)$.

Proof. Let $\tilde{\mathfrak{s}}$ denote the right hand side of 3.58. Since $[(\mathfrak{g}_c)_{\bar{\imath}}, (\mathfrak{g}_c)_{\bar{\jmath}}] \subseteq (\mathfrak{g}_c)_{\bar{\imath}+\bar{\jmath}}$ for $i, j \in \mathbb{Z}$ we obtain that $\tilde{\mathfrak{s}}$ is a subalgebra of $\tilde{\mathfrak{g}}$.

To prove that $\tilde{\mathfrak{g}}_c \subseteq \tilde{\mathfrak{s}}$ it suffices to show that $\tilde{\mathfrak{g}}_{\tilde{\alpha}} \subseteq \tilde{\mathfrak{s}}$ for $\tilde{\alpha} \in \tilde{R}^\times$. For this we let $\tilde{\alpha} = \pi(\alpha) + i\delta$ where $i \in \mathbb{Z}$, $\alpha \in R_{\bar{\imath}}$, $(\pi(\alpha), \pi(\alpha)) \neq 0$. Then $\tilde{\mathfrak{g}}_{\tilde{\alpha}} = \mathfrak{g}_{\bar{\imath},\pi(\alpha)} \otimes t^i$ which is contained in $(\mathfrak{g}_c)_{\bar{\imath}} \otimes t^i$ since, by Lemma 3.53, $\mathfrak{g}_{\bar{\imath},\pi(\alpha)}$ is contained in $\mathfrak{g}_c$. Thus, $\tilde{\mathfrak{g}}_c \subseteq \tilde{\mathfrak{s}}$.

To prove that $\tilde{\mathfrak{s}} \subseteq \tilde{\mathfrak{g}}_c$ we first show that $c \in \tilde{\mathfrak{g}}_c$. Indeed, since $\tilde{R}^\times$ is not empty we can choose an $\tilde{\epsilon} \in \tilde{R}^\times$. Then $\tilde{\epsilon} = \pi(\epsilon) + k\delta$ where $k \in \mathbb{Z}$, $\epsilon \in R_{\bar{k}}$, $(\pi(\epsilon), \pi(\epsilon)) \neq 0$. Then we also have that $-\epsilon \in R_{-\bar{k}}$ and furthermore $\mathfrak{g}_{\bar{k},\pi(\epsilon)}$ and $\mathfrak{g}_{-\bar{k},\pi(-\epsilon)}$ are paired, in a non-degenerate manner, by the form $(\cdot,\cdot)$ on $\mathfrak{g}$. Thus, we can choose $x \in \mathfrak{g}_{\bar{k},\pi(\epsilon)}$ and $y \in \mathfrak{g}_{-\bar{k},\pi(-\epsilon)}$ so that $(x, y) \neq 0$. Also, $x \otimes t^k$ is in $\tilde{\mathfrak{g}}_{\tilde{\epsilon}}$ and $y \otimes t^{-k}$ is in $\tilde{\mathfrak{g}}_{-\tilde{\epsilon}}$ and hence both of these elements belong to $\tilde{\mathfrak{g}}_c$. Similarly, since $x \in \mathfrak{g}_{\bar{k}+\bar{m},\pi(\epsilon)}$ and $y \in \mathfrak{g}_{-\bar{k}-\bar{m},\pi(-\epsilon)}$ we have $x \otimes t^{k+m} \in \tilde{\mathfrak{g}}_{\tilde{\epsilon}+m\delta}$ and $y \otimes t^{-k-m} \in \tilde{\mathfrak{g}}_{-\tilde{\epsilon}-m\delta}$ and so the elements $x \otimes t^{k+m}$, $y \otimes t^{-k-m}$ are in $\tilde{\mathfrak{g}}_c$. Thus $[x \otimes t^k, y \otimes t^{-k}] - [x \otimes t^{k+m}, y \otimes^{-k-m}]$ is in $\tilde{\mathfrak{g}}_c$. But this element equals $-m(x, y)c$, and so we get that $c \in \tilde{\mathfrak{g}}_c$ as desired.

For the proof of 3.58, it now only remains to show that $(\mathfrak{g}_c)_{\bar{\imath}} \otimes t^i \subseteq \tilde{\mathfrak{g}}_c$ for $i \in \mathbb{Z}$. But by Lemma 3.53, $(\mathfrak{g}_c)_{\bar{\imath}}$ is the sum of spaces of the form $\mathfrak{g}_{\bar{\imath},\pi(\alpha)}$, $\alpha \in R_{\bar{\imath}}$, $(\pi(\alpha), \pi(\alpha)) \neq 0$, and $[\mathfrak{g}_{\bar{\jmath},\pi(\beta)}, \mathfrak{g}_{\bar{l},\pi(\mu)}]$, for $j, l \in \mathbb{Z}$, $\bar{\jmath} + \bar{l} = \bar{\imath}$, $\beta \in R_{\bar{\jmath}}$, $\mu \in R_{\bar{l}}$, $(\pi(\beta), \pi(\beta)) \neq 0$, $(\pi(\mu), \pi(\mu)) \neq 0$. In the first case we have that $\mathfrak{g}_{\bar{\imath},\pi(\alpha)} \otimes t^i = \tilde{\mathfrak{g}}_{\pi(\alpha)+i\delta} \subseteq \tilde{\mathfrak{g}}_c$. In the second case we can assume that we have chosen j so that $j + l = i$. Then we have $[\mathfrak{g}_{\bar{\jmath},\pi(\beta)}, \mathfrak{g}_{\bar{l},\pi(\mu)}] \otimes t^i \subseteq [\mathfrak{g}_{\bar{\jmath},\pi(\beta)} \otimes t^j, \mathfrak{g}_{\bar{l},\pi(\mu)} \otimes t^l] + \mathbb{C}c$. But this is contained in $[\tilde{\mathfrak{g}}_{\pi(\beta)+j\delta}, \tilde{\mathfrak{g}}_{\pi(\mu)+l\delta}] + \mathbb{C}c \subseteq [\tilde{\mathfrak{g}}_c, \tilde{\mathfrak{g}}_c] + \mathbb{C}c \subseteq \tilde{\mathfrak{g}}_c$. So we have proved 3.58.

Since $\mathfrak{g}_c$ is perfect, $Z(\mathfrak{g}_c)$ is orthogonal to $\mathfrak{g}_c$. Thus, we have the inclusion "$\supseteq$" in 3.59. The reverse inclusion in 3.59 follows from 3.58 and the fact that any element of $Z(\tilde{\mathfrak{g}}_c)$ must commute with $(\mathfrak{g}_c)_{\bar{\imath}} \otimes t^j$ for all $j \in \mathbb{Z}$. Finally, to prove 3.60, observe that by 3.58 and 3.59 we have

$$\tilde{\mathfrak{g}}_c/Z(\tilde{\mathfrak{g}}_c) \cong \bigoplus_{i\in\mathbb{Z}} \big((\mathfrak{g}_c)_{\bar{\imath}}/Z(\mathfrak{g}_c)_{\bar{\imath}}\big) \otimes t^i \cong \bigoplus_{i\in\mathbb{Z}} (\mathfrak{g}_c/Z(\mathfrak{g}_c))_{\bar{\imath}} \otimes t^i = \mathrm{L}(\mathfrak{g}_c/Z(\mathfrak{g}_c), \sigma)$$

as vector spaces. Moreover, it is clear that this composite map is an isomorphism of algebras. □

We will actually not use 3.59 and 3.60 in what follows. However, the core modulo its center of an EALA plays a fundamental role in the structure theory of EALA's (see [BGK], [BGKN] and [AG]). Therefore 3.60 is of interest on its own.

Lemma 3.61. *Suppose that $\mathfrak{g}$ is tame and that $\tilde{R}^\times$ is not empty. Then $\tilde{\mathfrak{g}}$ is also tame.*

Proof. Let $\tilde{\mathfrak{c}}$ be the centralizer of $\tilde{\mathfrak{g}}_c$ in $\tilde{\mathfrak{g}}$. We must show that $\tilde{\mathfrak{c}} \subseteq \tilde{\mathfrak{g}}_c$.

Since $\tilde{\mathfrak{g}}_c$ is perfect, $\tilde{\mathfrak{c}}$ is orthogonal to $\tilde{\mathfrak{g}}_c$. Thus, $\tilde{\mathfrak{c}}$ is orthogonal to the central element c by 3.58. Hence $\tilde{\mathfrak{c}} \subseteq (\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{\imath}} \otimes t^i) \oplus \mathbb{C}c$. Now let $x \in \tilde{\mathfrak{c}}$. Then, $x = \sum_{i\in\mathbb{Z}} x_i \otimes t^i + rc$ for some $x_i \in \mathfrak{g}_{\bar{\imath}}$, $r \in \mathbb{C}$. But x commutes with $\tilde{\mathfrak{g}}_c$ and hence with $(\tilde{\mathfrak{g}}_c)_{\bar{\jmath}} \otimes t^j$ for all $j \in \mathbb{Z}$ by 3.58. Thus, for each i, x_i commutes with $(\mathfrak{g}_c)_{\bar{\jmath}}$ for all j and so $[x_i, \mathfrak{g}_c] = \{0\}$. Hence, since $\mathfrak{g}$ is tame, we have $x_i \in (\mathfrak{g}_c)_{\bar{\imath}}$ and so $x \in \tilde{\mathfrak{g}}_c$ by 3.58. □

We now want to combine all of our previous work to conclude that $\tilde{\mathfrak{g}}$ is a tame EALA provided that $\mathfrak{g}$ is tame and $\tilde{R}^\times$ is not empty. For this, we still need to see if EA5b holds for $\tilde{\mathfrak{g}}$. But, as we see in the next lemma, it is a general fact that if we

have a triple $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ of two Lie algebras and a form which satisfies EA1, EA2, EA3, EA4, EA5a and if $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ is tame, then $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ automatically satisfies EA5b. No doubt, this result is of independent interest. Of course, the core of such a triple, as well as the concept of tameness, are defined in the usual manner.

Lemma 3.62. *Let $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ be a triple consisting of two Lie algebras as well as a symmetric form and assume that EA1, EA2, EA3, EA4 and EA5a hold. Further, let $\mathfrak{g}_c$ be the subalgebra of $\mathfrak{g}$ generated by the non-isotropic root spaces and assume that $C_{\mathfrak{g}}(\mathfrak{g}_c) \subseteq \mathfrak{g}_c$. Then EA5b also holds and so $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ is a tame EALA.*

Proof. We use the notation from Section 1. It is clear that $\mathfrak{g}_c$ is perfect, that $C_{\mathfrak{g}}(\mathfrak{g}_c)$ is contained in the orthogonal complement to $\mathfrak{g}_c$, and that if δ is any root then the restriction of the form to the space $\mathfrak{g}_\delta + \mathfrak{g}_{-\delta}$ is non-degenerate. In order to show EA5b holds we let δ be any isotropic root of $\mathfrak{g}$.

If $\mathfrak{g}_\delta + \mathfrak{g}_{-\delta} \subseteq C_{\mathfrak{g}}(\mathfrak{g}_c)$ then it follows, since $C_{\mathfrak{g}}(\mathfrak{g}_c)$ is orthogonal to $\mathfrak{g}_c$ and since $C_{\mathfrak{g}}(\mathfrak{g}_c) \subseteq \mathfrak{g}_c$, that $\mathfrak{g}_\delta + \mathfrak{g}_{-\delta}$ is totally isotropic. This is impossible and so $\mathfrak{g}_\delta + \mathfrak{g}_{-\delta} \nsubseteq C_{\mathfrak{g}}(\mathfrak{g}_c)$. Thus, since the non-isotropic root spaces generate $\mathfrak{g}_c$, it follows that there exists a non-isotropic root $\alpha \in R$ such that either $[\mathfrak{g}_\alpha, \mathfrak{g}_\delta] \neq \{0\}$ or $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\delta}] \neq \{0\}$. If $[\mathfrak{g}_\alpha, \mathfrak{g}_\delta] \neq \{0\}$ then we get that $\alpha + \delta \in R$. If we have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\delta}] \neq \{0\}$ then we get that $-\delta + \alpha \in R$ and so its negative $\delta - \alpha$ is in R. In either case we get what we want. □

Combining the results of this section, we have the following theorem.

Theorem 3.63. *Let $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ be a tame extended affine Lie algebra, let m be a positive integer and let σ be an automorphism of $\mathfrak{g}$ satisfying*

A1. $\sigma^m = 1$
A2. $\sigma(\mathfrak{h}) = \mathfrak{h}$
A3. $(\sigma(x), \sigma(y)) = (x, y)$ *for all* $x, y \in \mathfrak{g}$
A4. *The centralizer of $\mathfrak{h}^\sigma$ in $\mathfrak{g}^\sigma$ equals $\mathfrak{h}^\sigma$.*

Let $\tilde{\mathfrak{g}} = \mathrm{Aff}(\mathfrak{g}, \sigma)$, $\tilde{\mathfrak{h}} = \mathfrak{h}^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$ and let $(\cdot,\cdot)$ be the form defined by 2.4 (restricted to $\mathrm{Aff}(\mathfrak{g}, \sigma)$). Let $\tilde{R}^\times$ be the set of nonisotropic roots $\tilde{\mathfrak{g}}$ relative to $\tilde{\mathfrak{h}}$. Then either $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ is a tame EALA or $\tilde{R}^\times$ is empty. Furthermore, $\tilde{R}^\times$ is nonempty if and only if

$$(\pi(\alpha), \pi(\alpha)) \neq 0 \text{ for some } \alpha \in R, \tag{3.64}$$

where $\pi(\alpha) = \frac{1}{m}\sum_{i=0}^{m-1} \sigma^i(\alpha)$.

Proof. The first statement follows from 3.28, 3.32, 3.33, 3.49 (i), 3.61 and 3.62. The second statement follows from 3.30. □

If $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ is a tame EALA, σ satisfies A1, A2 and A3, and m is a prime, the following corollary tells us that we can test whether or not $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ is a tame EALA using only information about the root system R of $\mathfrak{g}$ and the action of σ on R. This corollary follows immediately from the theorem and Proposition 3.25.

Corollary 3.65. *Let* $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ *be a tame EALA, let* m *be a positive integer and let* σ *be an automorphism of* $\mathfrak{g}$ *satisfying A1, A2 and A3. Let* $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ *be as in Theorem 3.63. If* $\pi(\alpha) \neq 0$ *for all nonzero* $\alpha \in R$ *and* $(\pi(\alpha), \pi(\alpha)) \neq 0$ *for some* $\alpha \in R$*, then* $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ *is a tame EALA. Moreover, the converse is true if* m *is a prime.*

Remark 3.66. Let $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ be a tame EALA, let σ be an automorphism of $\mathfrak{g}$ which satisfies A1, A2, A3 and A4, and suppose that $\widetilde{R}^\times$ is nonempty. Then $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ is a tame EALA. It is interesting to compare the type and nullity of $\mathfrak{g}$ with the type and nullity of $\tilde{\mathfrak{g}}$. Recall that the type of $\mathfrak{g}$ is the type of the finite root system $\bar{R}$, whereas the nullity ν of $\mathfrak{g}$ is $\dim_{\mathbb{R}} V^0$, where V^0 is the radical of the real span V of the roots of $\mathfrak{g}$.

(i) One can compute the type of $\tilde{\mathfrak{g}}$ by calculating the finite root system $\overline{\tilde{R}} = \bar{\pi}(\bar{R})$ using the formula $\bar{\pi}(\bar{\alpha}) = \frac{1}{m}\sum_{i=0}^{m-1} \bar{\sigma}^i(\bar{\alpha})$ (see 3.48).

(ii) The nullity $\tilde{\nu}$ of $\tilde{\mathfrak{g}}$ is given by

$$\tilde{\nu} = \dim_{\mathbb{R}}((V^0)^\sigma) + 1 \leq \nu + 1,$$

where $(V^0)^\sigma$ is the space of fixed points of σ acting on V^0 (see 3.41 and 3.45).

Corollary 3.67. *Let* $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ *be a tame EALA. Let* $\mathrm{Aff}(\mathfrak{g})$ *be the affinization of* $\mathfrak{g}$ *and let* $(\cdot,\cdot)$ *be the form defined by 2.4. Then, the triple*

$$(\mathrm{Aff}(\mathfrak{g}), \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, (\cdot,\cdot))$$

is a tame EALA. Moreover, the type of $\mathrm{Aff}(\mathfrak{g})$ *is the same as the type of* $\mathfrak{g}$*, and the nullity of* $\mathrm{Aff}(\mathfrak{g})$ *is one more than the nullity of* $\mathfrak{g}$*.*

Proof. Let $\sigma = \mathrm{id}_{\mathfrak{g}}$ and $m = 1$. Clearly A1, A2, A3 and A4 hold. Also $\pi(\alpha) = \alpha$ for $\alpha \in R$ and so 3.64 is clear. Thus, by Theorem 3.63, $\tilde{\mathfrak{g}} = \mathrm{Aff}(\mathfrak{g})$ is a tame EALA. Furthermore, by Remark 3.66 (i), $\overline{\tilde{R}} = \bar{R}$ and so $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ have the same type. Also, by Remark 3.66 (ii), we have $\tilde{\nu} = \nu + 1$. $\square$

We conclude this section with a brief discussion of degeneracy of EALA's. Recall from [G] that an EALA $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ is said to be *non-degenerate* if the dimension of the real vector space V^0 spanned by the isotropic roots R^0 of $\mathfrak{g}$ equals the dimension of the complex vector space spanned by R^0. The assumption that an EALA is nondegenerate simplifies the description of its structure. Thus, it is of interest to describe the extent to which affinization preserves degeneracy.

If W is a subset of a complex vector space, we let $W_{\mathbb{C}}$ denote the complex space spanned by W. Using this notation, we have that

$$\mathfrak{g} \text{ is non-degenerate if and only if } \dim_{\mathbb{C}}((V^0)_{\mathbb{C}}) = \dim_{\mathbb{R}}(V^0).$$

Otherwise said, $\mathfrak{g}$ is non-degenerate if and only if $\dim_{\mathbb{C}}((V^0)_{\mathbb{C}}) = \dim_{\mathbb{C}}(V^0 \otimes_{\mathbb{R}} \mathbb{C})$.

We now assume that we have an EALA $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ and an automorphism σ of $\mathfrak{g}$ which satisfies A1, A2, A3, and A4. Let $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{h}}$ be as in Theorem 3.63. Although $\tilde{\mathfrak{g}}$ is not in general an EALA, we nevertheless say that $\tilde{\mathfrak{g}}$ is *non-degenerate* if $\dim_{\mathbb{C}}((\tilde{V}^0)_{\mathbb{C}}) = \dim_{\mathbb{R}}(\tilde{V}^0)$.

Proposition 3.68. *Let $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ be a tame non-degenerate extended affine Lie algebra and let σ be an automorphism of $\mathfrak{g}$ which satisfies A1, A2, A3 and A4. Let $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ be as in Theorem 3.63. Then $\tilde{\mathfrak{g}}$ is non-degenerate. In particular, if $\tilde{\mathfrak{g}}$ is an EALA then $\tilde{\mathfrak{g}}$ is a non-degenerate EALA.*

Proof. We are assuming that

$$\dim_{\mathbb{C}}((V^0)_{\mathbb{C}}) = \dim_{\mathbb{R}}(V^0), \tag{3.69}$$

and we want to prove that $\dim_{\mathbb{C}}((\tilde{V}^0)_{\mathbb{C}}) = \dim_{\mathbb{R}}(\tilde{V}^0)$. Now by 3.46 we have that $\tilde{V}^0 = (V^0)^\sigma \oplus \mathbb{R}\delta$ and so $\dim_{\mathbb{R}}(\tilde{V}^0) = \dim_{\mathbb{R}}((V^0)^\sigma)+1$. But we also have $((V^0)^\sigma)_{\mathbb{C}} \cap \mathbb{C}\delta = \{0\}$ by the very definition of δ (see 3.16 and 3.17). Hence, $(\tilde{V}^0)_{\mathbb{C}} = ((V^0)^\sigma)_{\mathbb{C}} \oplus \mathbb{C}\delta$ and so $\dim_{\mathbb{C}}((\tilde{V}^0)_{\mathbb{C}}) = \dim_{\mathbb{C}}(((V^0)^\sigma)_{\mathbb{C}}) + 1$. Thus it is enough to show that

$$\dim_{\mathbb{C}}(((V^0)^\sigma)_{\mathbb{C}}) = \dim_{\mathbb{R}}((V^0)^\sigma). \tag{3.70}$$

Now 3.69 is equivalent to the statement that the natural $\mathbb{C}$-linear map $V^0 \otimes_{\mathbb{R}} \mathbb{C} \to (V^0)_{\mathbb{C}}$ is an isomorphism. Equivalently, 3.69 says that every subset of V^0 that is independent over $\mathbb{R}$ is independent over $\mathbb{C}$. This property is inherited by real subspaces of V^0 and so 3.70 holds. $\square$

§4 Examples

In this section, we describe three examples that illustrate the use of our main theorem to construct an EALA $\mathrm{Aff}(\mathfrak{g}, \sigma)$ starting from an EALA $\mathfrak{g}$ and an automorphism σ of finite order.

In the first example, $\mathfrak{g}$ is a toroidal Lie algebra coordinatized by the ring of commutative Laurent polynomials in ν variables.

Example 4.1. Let

$$\mathcal{A} = \mathbb{C}[t_1^{\pm 1}, \cdots, t_\nu^{\pm 1}]$$

be the ring of commutative Laurent polynomials in ν variables. In $\mathcal{A}$ we use the standard notation $t^{\boldsymbol{p}} = t_1^{p_1} \dots t_\nu^{p_\nu}$ for $\boldsymbol{p} = (p_1, \dots, p_\nu) \in \mathbb{Z}^\nu$. Thus $\mathcal{A}$ is a $\mathbb{Z}^\nu$-graded commutative associative algebra with $\mathcal{A}^{\boldsymbol{p}} = \mathbb{C}t^{\boldsymbol{p}}$ for $\boldsymbol{p} \in \mathbb{Z}^\nu$. We will make use of the $\mathbb{C}$-linear map $\epsilon : \mathcal{A} \to \mathbb{C}$ given by linear extension of

$$\epsilon(t^{\boldsymbol{p}}) = \begin{cases} 1 & \text{if } \boldsymbol{p} = 0 \\ 0 & \text{if } \boldsymbol{p} \neq 0 \end{cases}.$$

Let $\dot{\mathfrak{g}}$ be a finite dimensional simple complex Lie algebra. Let τ be an automorphism of $\dot{\mathfrak{g}}$ of period m. We fix a Cartan subalgebra $\mathfrak{k}$ of the nonzero reductive Lie algebra $\dot{\mathfrak{g}}^\tau$ (see [BM] and [P]). Then the centralizer $\dot{\mathfrak{h}}$ of $\mathfrak{k}$ in $\dot{\mathfrak{g}}$ is a Cartan subalgebra of $\dot{\mathfrak{g}}$ which is stable under τ (ibid). Clearly $\dot{\mathfrak{h}}^\tau = \mathfrak{k}$.

Based on this choice of $\dot{\mathfrak{g}}$ and $\dot{\mathfrak{h}}$, we construct a tame EALA $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ along the lines of Example 1.29 of Chapter 3 of [AABGP]. We recall how this goes.

First let $\mathcal{K} = \dot{\mathfrak{g}} \otimes \mathcal{A}$. Then, $\mathcal{K}$ is a $\mathbb{Z}^\nu$-graded Lie algebra with $\mathcal{K}^{\boldsymbol{p}} = \dot{\mathfrak{g}} \otimes \mathcal{A}^{\boldsymbol{p}}$. Let $(\cdot,\cdot)_{\mathcal{K}}$ be the unique bilinear form on $\mathcal{K}$ satisfying $(\dot{x} \otimes a, \dot{y} \otimes b)_{\mathcal{K}} = (\dot{x}, \dot{y})\epsilon(ab)$ for $\dot{x}, \dot{y} \in \dot{\mathfrak{g}}$, $a, b \in \mathcal{A}$, where $(\dot{x}, \dot{y})$ is the Killing form on $\dot{\mathfrak{g}}$. Then $(\cdot,\cdot)_{\mathcal{K}}$ is a nondegenerate invariant symmetric bilinear form on $\mathcal{K}$.

As vector spaces we let

$$\mathfrak{g} = \mathcal{K} \oplus \mathcal{C} \oplus \mathcal{D},$$

where $\mathcal{C} = \mathbb{C}c_1 \oplus \cdots \oplus \mathbb{C}c_\nu$ and $\mathcal{D} = \mathbb{C}d_1 \oplus \cdots \oplus \mathbb{C}d_\nu$ are ν-dimensional. The bracket on $\mathfrak{g}$ is defined so that

$$\begin{gathered}
[\mathfrak{g}, \mathcal{C}] = [\mathcal{D}, \mathcal{D}] = \{0\}, \\
[d_i, x] = p_i x \quad \text{for all } 1 \le i \le \nu \text{ and } x \in \mathcal{K}^{\boldsymbol{p}}, \\
[x, y] = [x, y]_{\mathcal{K}} + \sum_{i=1}^{\nu} ([d_i, x], y)_{\mathcal{K}} c_i \text{ for all } x, y \in \mathcal{K},
\end{gathered} \tag{4.2}$$

where $[\cdot,\cdot]_{\mathcal{K}}$ denotes the bracket on $\mathcal{K}$. The bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ is defined so that

$$\begin{gathered}
(\cdot,\cdot) \text{ extends } (\cdot,\cdot)_{\mathcal{K}}, \\
(\mathcal{C}, \mathcal{C}) = (\mathcal{D}, \mathcal{D}) = (\mathcal{C}, \mathcal{K}) = (\mathcal{D}, \mathcal{K}) = \{0\} \quad \text{and} \\
(c_i, d_j) = \delta_{ij}, \ i, j = 1, \ldots, \nu.
\end{gathered} \tag{4.3}$$

Finally, we set

$$\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathcal{C} \oplus \mathcal{D}$$

where we are identifying $\dot{\mathfrak{h}} = \dot{\mathfrak{h}} \otimes 1$. Then $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ is a tame EALA.

We now describe the automorphism σ of $\mathfrak{g}$ that we will use. First we fix $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_\nu) \in \mathbb{Z}^\nu$. Then $\boldsymbol{\mu}$ uniquely defines an automorphism $\mu \in Aut(\mathcal{A})$ satisfying $\mu(t^{\boldsymbol{p}}) = \zeta^{\boldsymbol{\mu} \cdot \boldsymbol{p}} t^{\boldsymbol{p}}$ where $\boldsymbol{\mu} \cdot \boldsymbol{p} := \mu_1 p_1 + \ldots + \mu_\nu p_\nu$. Both μ and τ admit natural extensions (that again we will denote by μ and τ) to $\mathrm{Aut}(\mathfrak{g})$ as follows:

$$\mu \text{ acts like } 1 \otimes \mu \text{ on } \dot{\mathfrak{g}} \otimes \mathcal{A}, \text{ and } \mu \text{ fixes } \mathcal{C} \text{ and } \mathcal{D} \text{ pointwise}$$

and

$$\tau \text{ acts like } \tau \otimes 1 \text{ on } \dot{\mathfrak{g}} \otimes \mathcal{A}, \text{ and } \tau \text{ fixes } \mathcal{C} \text{ and } \mathcal{D} \text{ pointwise.}$$

We set

$$\sigma := \tau\mu.$$

We claim that σ satisfies conditions A1 thru A4. Indeed A1 is clear since τ and μ commute. A2 holds because both τ and μ stabilize $\dot{\mathfrak{h}}$, $\mathcal{C}$ and $\mathcal{D}$. A3 follows from the fact that the Killing form on $\dot{\mathfrak{g}}$ is τ-invariant. To check A4, first note that

$$\mathfrak{g}^\sigma = \Big(\bigoplus_{\boldsymbol{p} \in \mathbb{Z}^\nu} \dot{\mathfrak{g}}_{-\overline{\boldsymbol{\mu} \cdot \boldsymbol{p}}} \otimes \mathcal{A}^{\boldsymbol{p}}\Big) \oplus \mathcal{C} \oplus \mathcal{D} \quad \text{and} \quad \mathfrak{h}^\sigma = (\mathfrak{k} \otimes 1) \oplus \mathcal{C} \oplus \mathcal{D}$$

If $\boldsymbol{p} \in \mathbb{Z}^\nu \setminus 0$ and $x \in \dot{\mathfrak{g}}$, then $[\mathcal{D}, x \otimes \mathcal{A}^{\boldsymbol{p}}] = \{0\}$ forces $x = 0$. As a consequence $C_{\dot{\mathfrak{g}}^\sigma}(\mathfrak{h}^\sigma)$ is contained in $(\mathfrak{g}_{\overline{0}} \otimes \mathcal{A}^0) \oplus \mathcal{C} \oplus \mathcal{D} = (\dot{\mathfrak{g}}^\tau \otimes 1) \oplus \mathcal{C} \oplus \mathcal{D}$. But $C_{\dot{\mathfrak{g}}^\tau}(\mathfrak{k}) = \mathfrak{k}$ (since $\mathfrak{k}$ is a Cartan subalgebra of $\dot{\mathfrak{g}}^\tau$), and therefore $C_{\mathfrak{g}^\sigma}(\mathfrak{h}^\sigma) = \mathfrak{h}^\sigma$ as prescribed by A4.

We now consider the triple $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$, where $\tilde{\mathfrak{g}} = \mathrm{Aff}(\mathfrak{g}, \sigma) = (\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{i}} \otimes t^i) \oplus \mathbb{C}c \oplus \mathbb{C}d$, $\tilde{\mathfrak{h}} = \mathfrak{h}^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $(\cdot,\cdot)$ is the restriction of the form 2.4. We want to next show that $\widetilde{R}^\times$ is nonempty. By Theorem 3.63, it will suffice to show that $(\pi(\alpha), \pi(\alpha)) \neq 0$ for some $\alpha \in R$. To see this, we identify $\dot{\mathfrak{h}}^*$ as a subspace of $\mathfrak{h}^*$ in the obvious way. Then the finite root system $\dot{R}$ of $\dot{\mathfrak{g}}$ with respect to $\dot{\mathfrak{h}}$ becomes identified with a subset of R. Now since $\{0\} \neq \mathfrak{k} \subset \dot{\mathfrak{g}}$ and $\dot{R}$ spans $\dot{\mathfrak{h}}^*$, there exists an $\alpha \in \dot{R}$ such that $\alpha \mid_{\mathfrak{k}} \neq 0$. But then $\pi(\alpha) \neq 0$ by definition of π. So $(\pi(\alpha), \pi(\alpha)) \neq 0$, since the Killing form restricted to the real span of $\dot{R}$ is definite. Thus, by Theorem 3.63, $\widetilde{R}$ is nonempty and hence $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ is a tame EALA.

It is interesting to note that the isomorphism class of the EALA $\tilde{\mathfrak{g}} = \mathrm{Aff}(\mathfrak{g}, \sigma)$ just constructed depends on τ but not on μ. This fact is a consequence of an "erasing" result that we will prove in the next paper in this series.

In our next example, $\mathfrak{g}$ is an EALA of type A_ℓ with a noncommutative coordinate algebra.

Example 4.4. Let $\mathbf{q} = (q_{ij})$ be a $\nu \times \nu$ complex matrix so that $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$. Let

$$\mathcal{A} = \mathbb{C}_{\mathbf{q}}[t_1^{\pm1}, \cdots, t_\nu^{\pm1}]$$

be the *quantum torus* determined by $\mathbf{q}$. Thus, by definition, $\mathcal{A}$ is the associative algebra generated by $t_1^{\pm1}, \ldots, t_\nu^{\pm1}$ subject to the relations $t_i {t_i}^{-1} = {t_i}^{-1} t_i = 1$ and $t_i t_j = q_{ij} t_j t_i$. In $\mathcal{A}$ we write $t^{\boldsymbol{p}} = t_1^{p_1} \ldots t_\nu^{p_\nu}$ for $\boldsymbol{p} = (p_1, \ldots, p_\nu) \in \mathbb{Z}^\nu$, in which case $\mathcal{A}$ is a $\mathbb{Z}^\nu$-graded associative algebra with $\mathcal{A}^{\boldsymbol{p}} = \mathbb{C}t^{\boldsymbol{p}}$ for $\boldsymbol{p} \in \mathbb{Z}^\nu$. We define $\epsilon : \mathcal{A} \to \mathbb{C}$ exactly as in the commutative case (Example 4.1).

The triple $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ that we use was introduced in [BGK]. We briefly recall the description of this EALA.

First of all let $\mathcal{K} = \mathfrak{sl}_{\ell+1}(\mathcal{A})$. That is, $\mathcal{K}$ is the Lie algebra of $(\ell+1) \times (\ell+1)$ matrices over $\mathcal{A}$ generated by the elementary matrices $ae_{i,j}$, $1 \leq i \neq j \leq \ell+1$, $a \in \mathcal{A}$. $\mathcal{K}$ has a unique $\mathbb{Z}^\nu$ grading so that $ae_{i,j} \in \mathcal{K}^{\boldsymbol{p}}$ for $1 \leq i \neq j \leq \ell+1$, $a \in \mathcal{A}^{\boldsymbol{p}}$. We define a nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)_{\mathcal{K}}$ on $\mathcal{K}$ by setting $(x,y)_{\mathcal{K}} = \epsilon(\mathrm{tr}(xy))$ for $x, y \in \mathcal{K}$. As vector spaces we let

$$\mathfrak{g} = \mathcal{K} \oplus \mathcal{C} \oplus \mathcal{D},$$

where $\mathcal{C} = \mathbb{C}c_1 \oplus \cdots \oplus \mathbb{C}c_\nu$ and $\mathcal{D} = \mathbb{C}d_1 \oplus \cdots \oplus \mathbb{C}d_\nu$ are ν-dimensional. As in Example 4.1, we define a bracket $[\,,\,]$ and a form $(\cdot,\cdot)$ on $\mathfrak{g}$ by (4.2) and (4.3). Finally, we set

$$\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathcal{C} \oplus \mathcal{D},$$

where $\dot{\mathfrak{h}} = \sum_{i=1}^{\ell} \mathbb{C}(e_{i,i} - e_{i+1,i+1})$. Then $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ is a tame EALA. The set R of roots of $\mathfrak{g}$ is given by

$$(4.5)\quad R = \{\sum_{k=1}^{\nu} n_k\delta_k \mid n_i \in \mathbb{Z}\} \cup \{\varepsilon_i - \varepsilon_j + \sum_{k=1}^{\nu} n_k\delta_k \mid 1 \le i \ne j \le \ell+1,\ n_i \in \mathbb{Z}\},$$

where $\varepsilon_i(h)$ is the i^{th} entry of h for $h \in \dot{\mathfrak{h}}$, ε_i is zero on $\mathcal{C}$ and $\mathcal{D}$, $\delta_i(d_j) = \delta_{i,j}$ and δ_i is zero on $\dot{\mathfrak{h}}$ and $\mathcal{C}$.

We now assume (in addition to our earlier assumptions) that $q_{ij} = \pm 1$ for all i, j. Then there exists an involution (antiautomorphism of period 2) $^-$ of $\mathcal{A}$ so that $\overline{t_i} = t_i$ for all i (see for example [AG, §2]). Since $\overline{t_1^{p_1} \dots t_\nu^{p_\nu}} = t_\nu^{p_\nu} \dots t_1^{p_1}$, the map $^-$ is called the *reversal involution* on $\mathcal{A}$. Using $^-$ we can define an involution $*$ on the associative algebra $M_{\ell+1}(\mathcal{A})$ by $(a_{ij})^* = (\overline{a_{\ell+2-j,\ell+2-i}})$. Then, we define a linear map $\sigma : \mathfrak{g} \to \mathfrak{g}$ by setting

$$\sigma(x) = -x^* \quad \text{for } x \in \mathcal{K},$$
$$\sigma|_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}} \quad \text{and} \quad \sigma|_{\mathcal{D}} = \mathrm{id}_{\mathcal{D}}.$$

Observe that if $x, y \in \mathcal{K}$, we have $\epsilon\big(\operatorname{tr}(yx)\big) = \epsilon\big(\operatorname{tr}(xy)\big)$ (see p. 366 of [BGK]), and so $(\sigma x, \sigma y)_{\mathcal{K}} = \epsilon\big(\operatorname{tr}((\sigma x)(\sigma y))\big) = \epsilon\big(\operatorname{tr}(x^* y^*)\big) = \epsilon\big(\operatorname{tr}((yx)^*)\big) = \epsilon\big(\operatorname{tr}(yx)\big) = \epsilon\big(\operatorname{tr}(xy)\big) = (x, y)_{\mathcal{K}}$. Using this fact is easy to check that σ is an automorphism of $\mathfrak{g}$ that preserves the form $(\cdot,\cdot)$. In fact it is clear that σ satisfies axioms A1, A2 and A3, where $m = 2$.

We next check A4 using Proposition 3.25. Indeed, one has

$$(4.6)\quad \begin{aligned} \sigma\big(\sum_{k=1}^{\nu} n_k\delta_k\big) &= \sum_{k=1}^{\nu} n_k\delta_k \quad \text{and} \\ \sigma\big(\varepsilon_i - \varepsilon_j + \sum_{k=1}^{\nu} n_k\delta_k\big) &= \varepsilon_{\ell+2-j} - \varepsilon_{\ell+2-i} + \sum_{k=1}^{\nu} n_k\delta_k. \end{aligned}$$

Thus, if α is a nonzero root in R, we have $\sigma(\alpha) \ne -\alpha$ and so $\pi(\alpha) \ne 0$. So A4 holds.

Once again we consider the triple $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ as in Theorem 3.63. Observe that $\pi(\varepsilon_1 - \varepsilon_2) = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_\ell - \varepsilon_{\ell+1})$ is nonisotropic. Hence, by Theorem 3.63, $\widetilde{R}$ is not empty and so $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ is a tame EALA.

Using 4.5, 4.6 and Remark 3.66, we can calculate the type and nullity of $\tilde{\mathfrak{g}} = \operatorname{Aff}(\mathfrak{g}, \sigma)$. Indeed, we have

$$\bar{R} = \{\bar{0}\} \cup \{\bar{\varepsilon}_i - \bar{\varepsilon}_j \mid 1 \le i \ne j \le \ell+1\}.$$

Thus, the finite root system $\overline{\widetilde{R}}$ associated with $\tilde{\mathfrak{g}}$ is

$$\overline{\widetilde{R}} = \bar{\pi}(\bar{R}) = \{\bar{0}\} \cup \{\frac{1}{2}(\bar{\varepsilon}_i - \bar{\varepsilon}_j + \bar{\varepsilon}_{\ell+2-j} - \bar{\varepsilon}_{\ell+2-i}) \mid 1 \le i \ne j \le \ell+1\}.$$

One easily sees that the type of this finite root system is C_p if $\ell = 2p-1$ and BC_p if $\ell = 2p$. Hence $\tilde{\mathfrak{g}}$ has type C_p if $\ell = 2p-1$ and type BC_p if $\ell = 2p$. Also, since all isotropic roots are fixed by σ, the nullity of $\tilde{\mathfrak{g}}$ is $\nu + 1$.

This example can be regarded as a noncommutative and higher nullity version of Kac's construction of the affine Lie algebra of type $A_\ell^{(2)}$ from the finite dimensional simple Lie algebra of type A_ℓ. (In fact this affine Lie algebra is alternately denoted by $C_{\boldsymbol{p}}^{(2)}$ if $\ell = 2p-1$ and $BC_p^{(2)}$ if $\ell = 2p$. See [MP].)

In the next example, we consider the case when the Lie algebra $\mathfrak{g}$ is an affine Kac-Moody Lie algebra and the automorphism σ is a diagram automorphism.

Example 4.7. Let $\mathfrak{g} = \mathfrak{g}(A) = \mathfrak{g}' \oplus \mathbb{C}d$ be the affine Kac–Moody Lie algebra constructed from an affine $(\ell+1)\times(\ell+1)$ generalized Cartan matrix A. We use the notation of [K2, Chapter 6]. Let $(\cdot,\cdot)$ be the normalized standard invariant form on $\mathfrak{g}$, and let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ used in the definition of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{h}, (\cdot,\cdot))$ is a tame EALA of nullity 1.

Let $\Pi = \{\alpha_0, \dots, \alpha_\ell\}$ be the root basis of $\mathfrak{g}$ and let $a_0, \dots, a_\ell$ denote the unique relatively prime positive integers so that $A[a_0, \dots, a_\ell]^t = 0$. Then, the lattice R^0 of isotropic roots is given by $R^0 = \mathbb{Z}\delta$, where $\delta = \sum_{i=0}^{\ell} a_i\alpha_i$. Consequently the real span V^0 of R^0 is given by $V^0 = \mathbb{R}\delta$.

Next let σ be an automorphism of period m of the GCM A. Thus, by definition, σ is a permutation of period m of the set $\{0, ..., \ell\}$ and σ satisfies $a_{\sigma i,\sigma j} = a_{i,j}$ for all i, j. We recall how to "extend" σ to an automorphism of $\mathfrak{g}$. Indeed it is shown in [FSS, §3.2] that there is a unique automorphism of $\mathfrak{g}$, which we also denote by σ, so that σ preserves the form $(\cdot,\cdot)$ on $\mathfrak{g}$ and

$$\sigma e_i = e_{\sigma i} \quad \text{and} \quad \sigma f_i = f_{\sigma i}$$

for all i. Moreover, σ stabilizes $\mathfrak{h}$ and has period m (ibid). We call the automorphism σ of $\mathfrak{g}$ the *diagram automorphism* associated with the automorphism σ of the GCM A.

Now σ acts on the set R of roots of $\mathfrak{g}$. Moreover since $\sigma e_i = e_{\sigma i}$, we have

$$\sigma\alpha_i = \alpha_{\sigma i}$$

for all i. Thus σ permutes the elements of Π.

It is clear that the diagram automorphism σ satisfies A1, A2 and A3. We now check A4. By Proposition 3.25, it is enough to show that $\pi(\alpha) \neq 0$ for all nonzero α in R. For this we can assume that α is positive. But σ permutes Π and stabilizes R. Thus $\sigma^i(\alpha)$ is a positive root for $i \geq 0$. Hence $\pi(\alpha) = \frac{1}{m}\sum_{i=0}^{m-1} \sigma^i\alpha$ is nonzero. So A4 holds.

We now consider the triple $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$, where $\tilde{\mathfrak{g}} = \mathrm{Aff}(\mathfrak{g},\sigma) = (\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{\imath}} \otimes t^i) \oplus \mathbb{C}\tilde{c} \oplus \mathbb{C}\tilde{d}$, $\tilde{\mathfrak{h}} = \mathfrak{h}^\sigma \oplus \mathbb{C}\tilde{c} \oplus \mathbb{C}\tilde{d}$ and $(\cdot,\cdot)$ is the restriction of the form 2.4. (Here we use

the notation $\tilde{c}$, $\tilde{d}$, and $\tilde{\delta}$ for the Lie algebra $\tilde{\mathfrak{g}}$, since we have already used c, d and δ for $\mathfrak{g}$.) We next want to decide when the set $\widetilde{R}^\times$ of nonisotropic roots of $\tilde{\mathfrak{g}}$ is non empty. For this purpose, we consider cases.

Suppose first that σ (or more precisely the group generated by σ) acts transitively on Π. Now from the uniqueness of the integers a_i it follows that $a_{\sigma i} = a_i$ for all i. Thus, by transitivity, all of the integers a_i are equal and so they are all 1. Thus, for each j, we have

$$\pi(\alpha_j) = \frac{1}{m}\sum_{i=0}^{m-1}\sigma^i\alpha_j = \frac{1}{m}\frac{m}{\ell+1}\sum_{i=0}^{\ell}\alpha_i = \frac{1}{\ell+1}\sum_{i=0}^{\ell}\alpha_i = \frac{1}{\ell+1}\delta.$$

Consequently, $\pi(\alpha_j)$ is isotropic for all j, and so $\pi(\alpha)$ is isotropic for all $\alpha \in R$. Hence, by Theorem 3.63, $\widetilde{R}^\times$ is empty.

Suppose next that σ does not act transitively on Π. Fix $j \in \{0, \ldots, \ell\}$. Then since σ does not act transitively, $\pi(\alpha_j)$ lies in the real span of a proper subset of Π. But, as we saw when verifying A4, $\pi(\alpha_j) \neq 0$. Thus $\pi(\alpha_j)$ is not isotropic. So by Theorem 3.63, $\widetilde{R}^\times$ is non empty and $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ is a tame EALA. Finally, to calculate the nullity of $\tilde{\mathfrak{g}}$, notice that $\sigma\delta$ is a positive root that generates R^0. Hence, $\sigma\delta = \delta$ and so $(V^0)^\sigma = V^0$ has real dimension 1. Thus, by Remark 3.66 (ii), $\tilde{\mathfrak{g}}$ has nullity 2.

We summarize the conclusions from this example in the following theorem.

Theorem 4.8. *Suppose that σ is a diagram automorphism of an affine Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$. Then σ satisfies axioms A1, A2, A3 and A4. Furthermore, let $\tilde{\mathfrak{g}} = \operatorname{Aff}(\mathfrak{g}, \sigma)$ and $\tilde{\mathfrak{h}} = \mathfrak{h}^\sigma \oplus \mathbb{C}\tilde{c} \oplus \mathbb{C}\tilde{d}$, and let $(\cdot,\cdot)$ be the restriction of the form 2.4. Then*

(i) *If σ acts transitively on Π, the set $\widetilde{R}^\times$ of nonisotropic roots of $\tilde{\mathfrak{g}}$ is empty.*
(ii) *If σ does not act transitively on Π, $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, (\cdot,\cdot))$ is a tame EALA of nullity 2.*

We note that case (i) in the theorem can only occur for one affine GCM, namely $A = A_\ell^{(1)}$, $\ell \geq 1$. (This follows from the classification of affine GCM's.) Furthermore, if $A = A_\ell^{(1)}$ and we label the roots of Π as in [K2, §6.1], then σ acts transitively on Π if and only if $\sigma = \tau^t$, where $\tau = (0, 1, \ldots, \ell)$ is the diagram rotation and t is relatively prime to $\ell + 1$.

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